ON THE MEAN CURVATURE OF MIXED TYPE SURFACES IN LORENTZ-MINKOWSKI SPACE

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Abstract:

In this paper, we study on mixed type surfaces in Lorentz-Minkowski space and give a theorem on the mean curvature of the surface with the degeneracy of first fundamental form.

1. INTRODUCTION

It is known that every closed surface in L^3must be of mixed type [1]. The first fundamental form of a mixed type surface is type-changing metric. Type-changing metrics are of interest both mathematics and theoretical physics [2].[3].

Definition 1.1 Let us denote by L^3 the Lorentz-Minkowski 3-space of signature $(++-)$. Consider an embedded surface S in L^3. The induced metric of S might be either positive definite, indefinite or degenerate. According to such properties, a point on the surface S is said to be spacelike, timelike or lightlike. If the spacelike, timelike and lightlike point sets are all non-empty, the surface S is said to be ***a mixed type surface*** [4]

In general, the mean curvature of such surfaces diverges. The graph of a smooth function $z = t(u, v)$ in Lorentz - Minkowski space L^3 gives a space-like (resp. time-like) surface if $B > 0$ $(resp.\ B < 0)$, where $B \coloneqq 1 - t_u{}^2 - t_v{}^2$

In this situation, the unit normal vector is given by

$$n = \frac{1}{\sqrt{|B|}}(1,\ t_u,\ t_v\ ),$$

and the mean curvature function is computed as

$$H = \frac{(t_u^2 - 1)t_{vv} - 2t_u t_v t_{uv} + (t_v^2 - 1)t_{uu}}{2|B(u,v)|^{3/2}}$$

which is unbounded around the set $\{B(u, v) = 0\}$, in general.
The main result of this paper is as follows:

Theorem : Let $f{:}\Sigma \rightarrow L^3$ be a mixed type surface given by
$f(u, v) = (u, v, t(u, v)\ )$

where $t(u, v)$ is a smooth function and $B{:}= EG - F^2 = 1 - t_u{}^2 - t_v{}^2$

If $B = 0$ then the mean curvature is given by

$$H = \begin{cases} \dfrac{t_{uu}}{2|B|^{3/2}}, & if\ t_{vv} = -\,t_{uu} \\ \dfrac{t_u^2 - t_v^2}{2|B|^{3/2}}, & if\ t_{vv} =\ t_{uu} \end{cases}$$

2. PRELIMINARIES

We denote by L^3 the Lorentz-Minkowski 3-space with the standard Lorentz metric

$$< x,x >_L = x_1^2 + x_2^2 - x_3^2\ ; x = \sum_{i=1}^{3} x_i e_i \in L^3$$

where $S \coloneqq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $e_1 \coloneqq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 \coloneqq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 \coloneqq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

A vector $x \in L^3$ $(x \neq 0)$ is said to be
spacelike if $< x,x >_L > 0$,
timelike if $< x,x >_L < 0$,
lightlike if $< x,x >_L = 0$.

2.1 Mixed Type Surfaces In this paper, a surface in $L3$ is defined to be an embedding $f{:}\Sigma \rightarrow L^3$ of a connected differentiable 2-manifold Σ. A point $p \in \Sigma$ is said to be a lightlike (resp. spacelike, timelike) point if the image $V_p \coloneqq df_p$ $(T_p\Sigma)$ of the tangent space $T_p\Sigma$ is lightlike (resp. spacelike, timelike) 2-subspace of $L3$. We denote by LD the set of lightlike points, $\Sigma+$ the set of spacelike points and $\Sigma-$the set of timelike points. If both $\Sigma+\ and \Sigma-$ are non-empty, the surface is called a mixed type surface.Denote by ds^2 the first fundamental form of f. ds^2 is the smooth metric onΣ defined by $ds^2 \coloneqq {}^*f <,>_L$. Then $p \in \Sigma$ is a lightlike point if and only if $(ds^2)_p$ degenerate as a symmetric bilinear

form on $T_p\Sigma$. Similarly $p \in \Sigma$ is a spacelike (resp. timelike) point if and only if $(ds^2)_p$ is positive definite (resp.indefinite). Take a local coordinate neighborhood $(U; u, v)$ of Σ.
Set
$f_u := df(\partial u)$ $f_v := df(\partial v)$. Then ds^2 is written as

$ds^2 = Edu^2 + 2Fdudv + Gdv^2$

$$where\ E := < f_u, f_u >_L, F := < f_u, f_v >_L, G := < f_v, f_v >_L$$

If $B := EG - F^2$ then a point $q \in U$ is lightlike (resp. spacelike, timelike) point if and only if $B(q) = 0$ $(resp.$ $B(q) > 0, B(q) < 0)$ holds. B is called as discriminant function.

2.2. Discriminant Function of A Mixed Type Surface We computed discriminant function of a mixed type surface in L^3 which is given by

$f{:}\Sigma \to L^3$, $f(u, v) = (u, v, t(u, v))$, where $t(u, v)$ is a smooth function For the surface we have

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$$f_u = (1,\ 0,\ t_u),\quad f_v = (0,\ 1,\ t_v)$$
$$f_{uu} = (0,\quad 0,\quad t_{uu}),\qquad f_{vv} = (0,\quad 0,\quad t_{vv})$$
$$f_{uv} = f_{vu} = (0,\quad 0,\quad 0)$$
$$E := < f_u, f_u >_L = 1 - t_u^2\ ,\ F := < f_u, f_v >_L = -t_u t_v,\ G := < f_v, f_v >_L = 1 - t_v^2$$
$$B := EG - F^2 = 1 - t_u^2 - t_v^2$$

3. PROOF OF MAIN RESULT

In this section we give the proof of the theorem in the introduction.

3.1 Proof of theorem Let $f{:}\Sigma \to L^3$ be a mixed type surface given by $f(u,v) = (u,\ v,\ t(u,v))$ where $t(u, v)$ is a smooth function and
$B := EG - F^2 = 1 - t_u^2 - t_v^2$.
If $B = 0 \Longrightarrow t_u^2 + t_v^2 = 1 \Longrightarrow t_u^2 - 1 = -t_v^2$ and $t_v^2 - 1 = -t_u^2$ (3.1)
$t_u^2 + t_v^2 = 1$ differentiate with respect to u we get $t_u t_{uu} = -t_v t_{uv}$ (3.2)
by substitution of (3.1) and (3.2) the numerator the mean curvature function is computed as
$(t_u^2 - 1)t_{vv} - 2t_u t_v t_{uv} + (t_v^2 - 1)t_{uu} = t_u^2 t_{uu} - t_v^2 t_{vv}$ and the mean curvature function is

$$H = \frac{t_u^2 t_{uu} - t_v^2 t_{vv}}{2|B(u,v)|^{3/2}}$$

Then we have

$$H = \begin{cases} \dfrac{t_{uu}}{2|B|^{3/2}}, & if\ t_{vv} = -\,t_{uu} \\ \dfrac{t_u^2 - t_v^2}{2|B|^{3/2}}, & if\ t_{vv} = \ t_{uu} \end{cases}$$

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