A STUDY OF COVERING SPACES THROUGH LATTICES

Dr.Pravanjan Kumar Rana[1*], Sarmad Hossain[2], Bhaskar Mandal[3]

[1]Associate Professor and Head, Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara,Kolkata-700118,WB,India*
[2&3]Ph.D., Research Scholar, Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, Kolkata-700118, WB,India,

Corresponding Author: -
pkranabgc@gmail.com,

Abstract.

Let C(X) denote the set of all covering spaces $(\tilde{X}, \tilde{x}, p)$ of (X, x) where (X, x) are path connected,locally path connected and semilocally simply connected pointed topological spaces.
In this paper we show that:
(i)(C(X), ≥) is a lattice and ($C^r(X)$, ≥) is a sublatice of ($C^r(X)$, ≥) without assuming π(X, x) is abelian, where C (X) is the set of all regular covering spaces of (X, x).
(ii)(C(X), ≥) is a modular, bounded and complete lattice when π(X, x) is abelian.

Keywords: *fundamental group, covering space, universal covering,regular covering,covering homomorphism, lattice.*

1. INTRODUCTION

Throughout the paper we assume that all the spaces (X, x) are path connected, locally path connected and semilocally simply connected pointed topological spaces and maps are base point preserving continuous maps. A covering space of a space (X, x) is a triple $(\tilde{X}, \tilde{x}, p)$ consisting of a pointed space (X, x) and a continuous surjective map $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ such that each point $x \in X$ has a path connected open neighborhood U such that each path component of $p^{-1}(U)$ is mapped homeomorphically onto U by p.

Let $(\widetilde{X_1}, \widetilde{x_1}, p_1)$ and $(\widetilde{X_2}, \widetilde{x_2}, p_2)$ be two covering spaces of (X, x). A homomorphism of $(\widetilde{X_1}, \widetilde{x_1}, p_1)$ into $(\widetilde{X_2}, \widetilde{x_2}, p_2)$ is a base point preserving continuous map $f :(\widetilde{X_1}, \widetilde{x_1}) \longrightarrow (\widetilde{X_2}, \widetilde{x_2})$ such that $p_2 f = p_1$. If in particular, f is a homeomorphism, then the coverings $(\widetilde{X_1}, \widetilde{x_1}, p_1)$ and $(\widetilde{X_2}, \widetilde{x_2}, p_2)$ are said to be isomorphic.

Let $C(X)$ denote the set of all covering spaces $(\tilde{X}, \tilde{x}, p)$ of (X, x). Then for each $(\tilde{X}, \tilde{x}, p) \in C(X)$, the map $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ induces a monomorphism $p_* : \pi(\tilde{X}, \tilde{x}) \rightarrow \pi(X, x)$ in the corresponding fundamental groups. The image group $H = p_*\pi((\tilde{X}, \tilde{x})$ depends on the choice of the base point $\tilde{x} \in p^{-1}(x)$.

Let $(S, \leq)$ be a partially ordered set (Poset) and a, b are any two elements of S. The least upper bound(lub) of $\{a, b\}$ in $(S, \leq)$, if it exists, is denoted by $a \vee b$. Similarly the greatest lower bound (glb) of $\{a, b\}$ in $(S, \leq)$, if it exists, is denoted by $a \wedge b$.

A poset $(S, \leq)$ is called an upper semi lattice if $a \vee b$ exists in S for all $a, b \in S$. Similarly a poset $(S, \leq)$ is called a lower semi lattice if $a \wedge b$ exists in S for all $a, b \in S$.

A poset $(L, \leq)$ is called a lattice if $a \vee b$ and $a \wedge b$ exits in L for all $a, b \in L$. Let $(L, \leq)$ be a lattice and L' be a nonempty subset of L such that $a \vee b$ and $a \wedge b$ exits in L' for all $a, b \in L'$,then $(L', \leq)$ is called a sublattice of $(L, \leq)$.

A lattice $(L, \leq)$ is called a modular lattice if for all $a, b, c \in L$, $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$ and is called a distributive lattice if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.

A lattice $(L, \leq)$ is called a complete lattice if every subsets of L have both a lub and a glb.

Let G be group and S be any subset of G. Write $< S > = \bigcap_T K$, where T is the collection of subgroups $K \subseteq G$ that contains S. Then $< S >$ is the smallest subgroup of G containing S and is called the subgroup generated by S. If G is an abelian group then every subgroup of G is normal, hence for any two subgroups A and B of G, $< A \cup B >= A + B$, their sum is a normal subgroup of G.

Lemma 1.1

Two covering spaces$(\widetilde{X_1}, \widetilde{x_1}, p_1)$ *and* $(\widetilde{X_2}, \widetilde{x_2}, p_2)$ such that $p_1(\widetilde{x_1}) = p_2(\widetilde{x_2}) = x$ are isomorphic if and only if the subgroups $p_1 * \pi(\widetilde{X_1}, \widetilde{x_1})$ and $p_2 * \pi(\widetilde{X_2}, \widetilde{x_2})$ belong to the same conjugacy class in $\pi(X, x)$.

Lemma 1.2

Given a subgroup H of $\pi(X, x)$,there exists a covering space $(\tilde{X}, \tilde{x}, p)$ of (X, x) such that $p * \pi(\tilde{X}, \tilde{x}) = H$.

Let $(\widetilde{X_1}, \widetilde{x_1}, p_1)$ *and* $(\widetilde{X_2}, \widetilde{x_2}, p_2) \in C(X)$. Define a binary relation ρ on $C(X)$ by $(\widetilde{X_1}, \widetilde{x_1}, p_1)\, \rho\, (\widetilde{X_2}, \widetilde{x_2}, p_2) \Leftrightarrow p_1 * \pi(\widetilde{X_1}, \widetilde{x_1}) = p_2 * \pi(\widetilde{X_2}, \widetilde{x_2})$. Then ρ is an equivalence relation.

Let $C(X)/\rho$ denote the set of all ρ equivalence classes $(\tilde{X}, \tilde{x}, p)\rho$ of the coverings of (X, x). Define $'\geq'$ on $C(X)/\rho$ by $(\widetilde{X_1}, \widetilde{x_1}, p_1)\, \rho \geq (\widetilde{X_2}, \widetilde{x_2}, p_2)\rho \Leftrightarrow p_1 * \pi(\widetilde{X_1}, \widetilde{x_1}) \subseteq p_2 * \pi(\widetilde{X_2}, \widetilde{x_2})$.

Proposition 1.3

$'\geq'$ is a partial order relation on $C(X)/\rho$.

Proof : As the relation $'\geq'$ on $C(X)/\rho$ is determined in terms of set inclusion, it follows that $'\geq'$ is a partial order relation on $C(X)/\rho$.

Theorem 1.4

The partially ordered set $(C(X)/\rho, \geq)$ is a semilattice.

Proof : Let $(\widetilde{X_1}, \widetilde{x_1}, p_1)\, \rho$, $(\widetilde{X_2}, \widetilde{x_2}, p_2)\rho \in C(X)/\rho$. Then $p_1 * \pi(\widetilde{X_1}, \widetilde{x_1})$ and $p_2{*}\pi(\widetilde{X_2}, \widetilde{x_2})$ are subgroups of $\pi(X, x)$. Let $A = p_1{*}\, \pi(\widetilde{X_1}, \widetilde{x_1}) \cap p_2{*}\pi(\widetilde{X_2}, \widetilde{x_2})$. Then A is a subgroup of $\pi(X, x)$. Hence by **Lamma 1.2**, we find a covering space $(\tilde{X}, \tilde{x}, p) \in C(X)$ such that $p * \pi(\tilde{X}, \tilde{x}) = A$.

Then $p{*}\pi(\tilde{X}, \tilde{x}) \subseteq p_1{*}\, \pi(\widetilde{X_1}, \widetilde{x_1}) \Leftrightarrow (\tilde{X}, \tilde{x}, p)\rho \geq (\widetilde{X_1}, \widetilde{x_1}, p_1)\rho$. Again, $p{*}\pi(\tilde{X}, \tilde{x}) \subseteq p_2{*}\pi(\widetilde{X_2}, \widetilde{x_2}) \Leftrightarrow (\tilde{X}, \tilde{x}, p)\rho \geq (\widetilde{X_2}, \widetilde{x_2}, p_2)\rho$. Consequently, $(\tilde{X}, \tilde{x}, p)\rho$ is an upper bound of $(\widetilde{X_1}, \widetilde{x_1}, p_1)\rho$ and $(\widetilde{X_2}, \widetilde{x_2}, p_2)\rho$. We claim that $(\tilde{X}, \tilde{x}, p)\rho$ is the lub of $(\widetilde{X_1}, \widetilde{x_1}, p_1)\rho$ and $(\widetilde{X_2}, \widetilde{x_2}, p_2)\rho$. Let $(\tilde{X}, \tilde{x}, p')\rho \geq (\widetilde{X_1}, \widetilde{x_1}, p_1)\, \rho$ and $(\tilde{X}, \tilde{x}, p')\rho \geq (\widetilde{X_2}, \widetilde{x_2}, p_2)\rho$ in $C(X)/\rho$. Then $p'{*}\pi(\tilde{X}', \tilde{x}') \subseteq p_1{*}\pi(\widetilde{X_1}, \widetilde{x_1})$ and $p'{*}\pi(\tilde{X}', \tilde{x}') \subseteq p_2{*}\pi(\widetilde{X_2}, \widetilde{x_2})$. Consequently, $p'{*}\pi(\tilde{X}', \tilde{x}'') \subseteq p_1{*}\pi(\widetilde{X_1}, \widetilde{x_1}) \cap p_2{*}\pi(\widetilde{X_2}, \widetilde{x_2}) = A = p * \pi(\tilde{X}, \tilde{x})$. Hence $(\tilde{X}', \tilde{x}', p')\rho \geq (\tilde{X}, \tilde{x}, p)\rho$.

We now define $'\vee'$ on $C(X)/\rho$ by the rule $(\widetilde{X_1}, \widetilde{x_1}, p_1)\rho \vee (\widetilde{X_2}, \widetilde{x_2}, p_2)\rho = (\tilde{X}, \tilde{x}, p)\rho$ (the latter is determined as above). Consequently the partially ordered set $(C(X)/\rho, \geq)$ is a semilattice.

Theorem 1.5

Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Then $(C(X), \geq)$ is a lattice.

Proof : We now consider covering spaces of (X, x). As $\pi(X, x)$ is abelian,

two subgroups of $\pi(X, x)$ are conjugate iff they are equal. Consequently, two covering spaces of (X, x) are isomorphic iff they correspond to the same subgroup of $\pi(X, x)$, by lemma 1.1. Let $(\widetilde{X_1},\widetilde{x_1},p_1)$, $(\widetilde{X_2},\widetilde{x_2}, p_2) \in C(X)$.
Define $(\widetilde{X_1},\widetilde{x_1},p_1) \geq (\widetilde{X_2},\widetilde{x_2}, p_2) \Leftrightarrow p_1 * \pi(\widetilde{X_1},\widetilde{x_1}) \subseteq p_2 * \pi(\widetilde{X_2},\widetilde{x_2})$. Then $'\geq'$ is a partial order relation and $(C(X), \vee)$ is an upper semilattice by **theorem 1.4.**
Again suppose $A = p_1*\pi(\widetilde{X_1},\widetilde{x_1})$ and $B = p_2*\pi(\widetilde{X_2},\widetilde{x_2})$ for some $(\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\widetilde{X_2},\widetilde{x_2}, p_2) \in C(X)$. As $\pi(X, x)$ is abelian, the subgroups A and B of $\pi(X, x)$ are also abelian. Hence their sum $A+B$ is also a subgroup of $\pi(X, x)$ such that $A \subseteq A + B$ and $B \subseteq A + B$. Then by **Lemma 1.2,** there exists a covering space $(\tilde{X},\tilde{x}, p)$ of (X, x) such that $p * \pi((\tilde{X},\tilde{x}) = A + B$. Now $A \subseteq A + B \Rightarrow p_1 * \pi(\widetilde{X_1},\widetilde{x_1}) \subseteq p * \pi(X, x) \Rightarrow (\widetilde{X_1},\widetilde{x_1}, p_1) \geq (\tilde{X},\tilde{x}, p)$.Similarly $B \subseteq A + B \Rightarrow (\widetilde{X_2},\widetilde{x_2}, p_2) \geq (\tilde{X},\tilde{x}, p)$.Thus $(\tilde{X},\tilde{x}, p)$ is a lower bound of $(\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\widetilde{X_2},\widetilde{x_2}, p_2)$.We claim that it is their glb. To prove this, let $(\tilde{X}',\tilde{x}', p') \in C(X)$ be such that $(\widetilde{X_1},\widetilde{x_1}, p_1) \geq (\tilde{X}',\tilde{x}', p')$ and $(\widetilde{X_2},\widetilde{x_2}, p_2) \geq (\tilde{X}',\tilde{x}', p')$ .Then $A \subseteq p' * \pi(\tilde{X}',\tilde{x}') = D$. Similarly $B \subseteq D$. Consequently $A + B \subseteq D$ and this implies $(\tilde{X},\tilde{x}, p) \geq (\tilde{X}',\tilde{x}', p')$. Define $'\wedge'$ on $C(X)$ by $(\widetilde{X_1},\widetilde{x_1}, p_1) \wedge (\widetilde{X_2},\widetilde{x_2}, p_2) = (\tilde{X},\tilde{x}, p)$ .Thus $(C(X), \wedge)$ is a lower semilattice.
Hence $(C(X), \geq)$ is a lattice.

Next we show that $(C(X), \geq)$ is a lattice without assuming that $\pi(X, x)$ is abelian.

Theorem 1.6

We show that $(C(X), \geq)$ is a lattice, without assuming that $\pi(X, x)$ is abelian.

Proof : Let $(\widetilde{X_1},\widetilde{x_1},p_1)$, $(\widetilde{X_2},\widetilde{x_2},p_2) \in C(X)$. Define $(\widetilde{X_1},\widetilde{x_1},p_1) \geq (\widetilde{X_2},\widetilde{x_2},p_2) \Leftrightarrow p_1*\pi(\widetilde{X_1},\widetilde{x_1}) \subseteq p_2*\pi(\widetilde{X_2},\widetilde{x_2})$. Then '$\geq$' is a partial order relation and $(C(X), \vee)$ is an upper semilattice by **theorem 1.4**.
Again suppose $A = p_1*\pi(\widetilde{X_1},\widetilde{x_1})$ and $B = p_2*\pi(\widetilde{X_2},\widetilde{x_2})$ for some $(\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\widetilde{X_2},\widetilde{x_2}, p_2) \in C(X)$. Let $S = A \cup B$.
Then $< S > = \bigcap_T K$, where T is the collection of subgroups $K \subseteq \pi(X, x)$ that contains both A and B. It is clear that $A \subseteq < S >$ and $B \subseteq < S >$. Then by **Lemma 1.2**, there exists a covering space $(\tilde{X}',\tilde{x}', p')$ of (X, x) such that $p' * \pi(\tilde{X}',\tilde{x}') = < S > = < A \cup B >$. Now $A \subseteq < S > \Rightarrow p_1 * \pi(\widetilde{X_1},\widetilde{x_1}) \subseteq p' * \pi(\tilde{X}',\tilde{x}') \Rightarrow (\widetilde{X_1},\widetilde{x_1}, p_1) \geq (\tilde{X}',\tilde{x}',p')$.
Similarly $B \subseteq < S > \Rightarrow (\widetilde{X_2},\widetilde{x_2},p_2) \geq (\tilde{X}',\tilde{x}', p')$. Thus $(\tilde{X}',\tilde{x}', p')$ is a lower bound of $(\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\widetilde{X_2},\widetilde{x_2},p_2)$. We claim that it is their glb. To prove this, let $(\tilde{X}'',\tilde{x}'', p'') \in C(X)$ be such that $(\widetilde{X_1},\widetilde{x_1}, p_1) \geq (\tilde{X}'',\tilde{x}'', p'')$ and $(\widetilde{X_2},\widetilde{x_2},p_2) \geq (\tilde{X}'',\tilde{x}'', p'')$.
Then $A \subseteq p'' * \pi(\tilde{X}'',\tilde{x}'')$. Similarly $B \subseteq p'' * \pi(\tilde{X}'',\tilde{x}'')$. Consequently $< S > = <A\cup B> \subseteq p''* \pi(\tilde{X}'',\tilde{x}'')$ by definition of $<S>$ and this implies $(\tilde{X}',\tilde{x}', p') \geq (\tilde{X}'',\tilde{x}'',p'')$. Define $'\wedge'$ on $C(X)$ by $(\widetilde{X_1},\widetilde{x_1}, p_1) \wedge (\widetilde{X_2},\widetilde{x_2},p_2) = (\tilde{X}',\tilde{x}', p')$. Thus $(C(X), \wedge)$ is a lower semilattice. Hence $(C(X), \geq)$ is a lattice.

In the section 2, if $C^r(X)$ be the collection of all regular covering spaces of (X, x), then we will show that $(C^r(X), \geq)$ is a sublattice of $(C(X), \geq)$. Finally, assuming $\pi(X, x)$ is abelian we will show that $(C(X), \geq)$ is a modular, bounded and complete lattice.

Theorem 2.1:

$(C^r(X), \geq)$ is a sublattice of $(C(X), \geq)$, where $C^r(X)$ be the collection of all regular covering spaces of (X, x).

Proof : Here $C^r(X) = \{(\tilde{X},\tilde{x}, p): (\tilde{X},\tilde{x}, p)$ is a regular covering space of $(X, x)\}$. As universal covering is regular, so $C^r(X)$ is a nonempty subset of $C(X)$. So it is enough to show that $(C^r(X), \geq)$ is a lattice that is $(C^r(X), \vee)$ is an upper semilattice and $(C^r(X), \vee)$ is a lower semilattice. Let $(\widetilde{X_1},\widetilde{x_1}, p_1),(\widetilde{X_2},\widetilde{x_2},p_2) \in C^r(X)$. Then $H_1 = p_1 * \pi(\widetilde{X_1},\widetilde{x_1})$ and $H_2 = p_2 * \pi(\widetilde{X_2},\widetilde{x_2})$ are normal subgroups of $\pi(X, x)$. Let $H = p_1 * \pi(\widetilde{X_1},\widetilde{x_1}) \cap p_2 * \pi(\widetilde{X_2},\widetilde{x_2})$. Then H is a (normal) subgroup of $\pi(X, x)$. Hence by **Lamma 1.2**, we find a covering space $(\tilde{X},\tilde{x}, p)$ of (X, x) such that $p * \pi(\tilde{X},\tilde{x}) = H$. As H is a normal subgroup of $\pi(X, x)$, $(\tilde{X},\tilde{x}, p) \in C^r(X)$. Then $p*\pi(\tilde{X},\tilde{x}) \subseteq p_1 * \pi(\widetilde{X_1},\widetilde{x_1}) \Leftrightarrow (\tilde{X},\tilde{x}, p) \geq (\widetilde{X_1},\widetilde{x_1}, p_1)$.
Again, $p * \pi(\tilde{X},\tilde{x}) \subseteq p_2 *\pi(\widetilde{X_2},\widetilde{x_2}) \Leftrightarrow (\tilde{X},\tilde{x}, p) \geq (\widetilde{X_2},\widetilde{x_2}, p_2)$. Consequently, $(\tilde{X},\tilde{x}, p)$ is an upper bound of $(\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\widetilde{X_2},\widetilde{x_2}, p_2)$. We claim that $(\tilde{X},\tilde{x}, p)$ is the lub of $(\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\widetilde{X_2},\widetilde{x_2}, p_2)$. Let $(\tilde{X}',\tilde{x}', p') \geq (\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\tilde{X}',\tilde{x}', p') \geq (\widetilde{X_2},\widetilde{x_2}, p_2)$ in $C^r(X)$.Then $p' * \pi(\tilde{X}',\tilde{x}') \subseteq p_1 * \pi(\widetilde{X_1},\widetilde{x_1})$ and $p'*\pi(\tilde{X}',\tilde{x}') \subseteq p_2*\pi(\widetilde{X_2},\widetilde{x_2})$. Consequently, $p' * \pi(\tilde{X}',\tilde{x}') \subseteq p_1* \pi(\widetilde{X_1},\widetilde{x_1}) \cap p_2 * \pi(\widetilde{X_2},\widetilde{x_2}) = H = p * \pi(\tilde{X},\tilde{x})$. Hence $(\tilde{X}',\tilde{x}', p') \geq (\tilde{X},\tilde{x}, p)$. We now define $'\vee'$ on $C^r(X)$ by the rule $(\widetilde{X_1},\widetilde{x_1}, p_1) \vee (\widetilde{X_2},\widetilde{x_2},p_2) = (\tilde{X},\tilde{x}, p)$. Consequently the partially ordered set $(C^r(X), \vee)$ is an upper semilattice.
Again, as H_1 and H_2 are normal subgroups of $\pi(X, x)$, hence $< H_1\cup H_2 > = H_1 + H_2$ is a (normal)subgroup of $\pi(X, x)$ such that $H_1 \subseteq H_1 + H_2$ and $H_2 \subseteq H_1 + H_2$. Then by **Lemma 1.2,** there exists a covering space $(\tilde{Y},\tilde{y}, p)$ of (X, x) such that $p*\pi(\tilde{Y},\tilde{y}) = H_1 + H_2$. As $H_1 + H_2$ is a normal subgroup of $\pi(X, x)$, $(\tilde{Y},\tilde{y}, p) \in C^r(X)$. Now $H_1 \subseteq H_1 + H_2 \Rightarrow p_1* \pi(\widetilde{X_1},\widetilde{x_1}) \subseteq p * \pi(\tilde{Y},\tilde{y}) \Rightarrow (\widetilde{X_1},\widetilde{x_1}, p_1) \geq (\tilde{Y},\tilde{y}, p)$. Similarly $H_2 \subseteq H_1 + H_2 \Rightarrow (\widetilde{X_2},\widetilde{x_2}, p_2) \geq (\tilde{Y},\tilde{y}, p)$.
Thus $(\tilde{Y},\tilde{y}, p)$ is a lower bound of $(\widetilde{X_1},\widetilde{x_1}, p_1)$ and $(\widetilde{X_2},\widetilde{x_2},p_2)$.We claim that it is their glb. To prove this, let $(\tilde{Y}',\tilde{y}', p') \in C^r(X)$ be such that $(\widetilde{X_1},\widetilde{x_1}, p_1) \geq (\tilde{Y}',\tilde{y}', p')$ and $(\widetilde{X_2},\widetilde{x_2}, p_2) \geq (\tilde{Y}',\tilde{y}', p')$. Then $H_1 \subseteq p' * \pi(\tilde{Y}',\tilde{x}')$. Similarly $H_2 \subseteq p' * \pi(\tilde{Y}',\tilde{x}')$. Consequently $H_1+H_2 \subseteq p'*\pi(\tilde{Y}',\tilde{x}')$ and this implies $(\tilde{Y},\tilde{y}, p) \geq (\tilde{Y}',\tilde{y}', p')$. Define $'\wedge'$ on $C^r(X)$ by $(\widetilde{X_1},\widetilde{x_1}, p_1) \wedge (\widetilde{X_2},\widetilde{x_2},p_2) = (\tilde{Y},\tilde{y}, p)$. Thus $(C^r(X), \wedge)$ is a lower semilattice. Hence $(C^r(X), \geq)$ is a lattice. Hence $(C^r(X), \geq)$ is a sublattice of $(C(X), \geq)$.

Theorem 2.2

Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Then $(C(X), \geq)$ is a modular lattice.

Proof : By **theorem 1.5**, $(C(X), \geq)$ is a lattice. we need to show that it is modular. Let $(\widetilde{X_1},\widetilde{x_1}, p_1)$, $(\widetilde{X_2},\widetilde{x_2},p_2)$,

$(\widetilde{X_3}, \widetilde{x_3}, p_3) \in C(X)$ be such that $(\widetilde{X_1}, \widetilde{x_1}, p_1) \geq (\widetilde{X_3}, \widetilde{x_3}, p_3)$.We have to show that $(\widetilde{X_1}, \widetilde{x_1}, p_1) \vee ((\widetilde{X_2}, \widetilde{x_2}, p_2) \wedge (\widetilde{X_3}, \widetilde{x_3}, p_3)) = ((\widetilde{X_1}, \widetilde{x_1}, p_1) \vee (\widetilde{X_2}, \widetilde{x_2}, p_2)) \wedge (\widetilde{X_3}, \widetilde{x_3}, p_3)$. Let $A = p_1 * \pi(\widetilde{X_1}, \widetilde{x_1})$, $B = p_2 * \pi(\widetilde{X_2}, \widetilde{x_2})$, $C = p_3 * \pi(\widetilde{X_3}, \widetilde{x_3})$. By definition of '$\geq$', it is enough to show that $A + (B \cap C) = (A + B) \cap C$ as $(A + B)$, $(B \cap C)$, $A + (B \cap C)$ and $(A + B) \cap C$ are all (normal) subgroups of $\pi(X, x)$, as $\pi(X, x)$ is abelian. Now $(\widetilde{X_1}, \widetilde{x_1}, p_1) \geq (\widetilde{X_3}, \widetilde{x_3}, p_3)$ implies $A \subseteq C$ which implies $A + (B \cap C) \subseteq (A + B) \cap C$. So, we have to show that $(A + B) \cap C \subseteq A + (B \cap C)$. Let $t \in (A + B) \cap C$. Then $t = a + b = c$ for $a \in A$, $b \in B$ and $c \in C$. Thus $t - a = b = c - a \in B \cap C$ as $A \subseteq C$ and $t = a + b \in A + (B \cap C)$. Consequently $(A + B) \cap C \subseteq A + (B \cap C)$. Hence $A + (B \cap C) = (A + B) \cap C$. Thus $(C(X), \geq)$ is a modular lattice.

Theorem 2.3
Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Then $(C(X), \geq)$ is a bounded lattice.
Proof : Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Now $(C(X), \geq)$ has the top element, the universal covering space and has the bottom element, the trivial covering space. Hence $(C(X), \geq)$ is a bounded lattice.

Theorem 2.4
Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Then $(C(X), \geq)$ is a complete lattice.
Proof : Let (X, x) be a space such that its fundamental group $\pi(X, x)$ is abelian. Let $S = \{(\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) : (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) \in C(X), \alpha \in I$ (an indexing set)$\}$ be a subset of $(C(X), \geq)$. Let $A_\alpha = p_\alpha * \pi(\widetilde{X_\alpha}, \widetilde{x_\alpha})$. Then A_α are (abelian and normal) subgroups of the abelian group $\pi(X, x)$. Let $A = \bigcap_{\alpha \in I} p_\alpha * \pi(\widetilde{X_\alpha}, \widetilde{x_\alpha})$. Then A is a (abelian and normal) subgroup of $\pi(X, x)$. Hence by **Lamma 1.2**, we find a covering space $(\tilde{X}, \tilde{x}, p) \in C(X)$ such that $p * \pi(\tilde{X}, \tilde{x}) = A$. Now $p * \pi(\tilde{X}, \tilde{x}) \subseteq p_\alpha * \pi(\widetilde{X_\alpha}, \widetilde{x_\alpha}) \Leftrightarrow (\tilde{X}, \tilde{x}, p) \geq (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha)$. Consequently, $(\tilde{X}, \tilde{x}, p)$ is an upper bound of $(\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha)$. We claim that $(\tilde{X}, \tilde{x}, p)$ is the lub of $(\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha)$. Let $(\tilde{X}', \tilde{x}', p') \geq (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha)$ in $C(X)$. Then $p' * \pi(\tilde{X}', \tilde{x}') \subseteq p_\alpha * \pi(\widetilde{X_\alpha}, \widetilde{x_\alpha})$ for every $\alpha \in I$. Consequently, $p' * \pi(\tilde{X}', \tilde{x}') \subseteq \bigcap_{\alpha \in I} p_\alpha * \pi(\widetilde{X_\alpha}, \widetilde{x_\alpha}) = A = p * \pi(\tilde{X}, \tilde{x})$.
Hence $(\tilde{X}', \tilde{x}', p') \geq (\tilde{X}, \tilde{x}, p)$. Hence $(\tilde{X}, \tilde{x}, p)$ is the lub of $S = \{(\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) : (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) \in C(X), \alpha \in I\}$.
That is $(\tilde{X}, \tilde{x}, p) = \bigvee_{\alpha \in I} (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha)$.
Now, let $B = < \bigcup_{\alpha \in I} A_\alpha >$.Then $B = \sum_{\alpha \in I} A_\alpha$ A_α, as A_α are (abelian and normal) subgroups of the abelian group $\pi(X, x)$. Hence B is a (abelian and normal) subgroup of $\pi(X, x)$. Hence by **Lamma 1.2,** we find a covering space $(\tilde{Y}, \tilde{y}, p) \in C(X)$ such that $p * \pi(\tilde{Y}, \tilde{y}) = B$. Now each $A_\alpha \subseteq B$ implies $p_\alpha * \pi(\widetilde{X_\alpha}, \widetilde{x_\alpha}) \subseteq p * \pi(\tilde{Y}, \tilde{y})$.Thus $(\tilde{Y}, \tilde{y}, p)$ is a lower bound of $S = \{(\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) : (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) \in C(X), \alpha \in I\}$. We claim that it is their glb. To prove this, let $(\tilde{Y}', \tilde{y}', p') \in C(X)$ be such that $\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) \geq (\tilde{Y}', \tilde{y}', p')$. Then $A_\alpha \subseteq p' * \pi(\tilde{Y}', \tilde{y}')$ and this implies $(\tilde{Y}, \tilde{y}, p) \geq (\tilde{Y}', \tilde{y}', p')$. Thus $(\tilde{Y}, \tilde{y}, p)$ is the glb of $S = \{(\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) : (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha) \in C(X), \alpha \in I\}$. That is $(\tilde{Y}, \tilde{y}, p) = \bigwedge_{\alpha \in I} (\widetilde{X_\alpha}, \widetilde{x_\alpha}, p_\alpha)$. Hence $(C(X), \geq)$ is a complete lattice.

In the section 3 we give some applications and examples.
Theorem 3.1
Let $(C(X), \geq)$ be a complete lattice and $f : (C(X), \geq) \to (C(X), \geq)$ be an isotone function. Then $f((\tilde{X}, \tilde{x}, p)) = (\tilde{X}, \tilde{x}, p)$ for some $(\tilde{X}, \tilde{x}, p) \in C(X)$.
Proof :Let $S = \{(\tilde{X}', \tilde{x}', p') \in C(X) : (\tilde{X}', \tilde{x}', p') \geq f((\tilde{X}', \tilde{x}', p'))\}$. As S is a subset of the complete lattice $(C(X),$), lub and glb of S exist. Define $(\tilde{X}, \tilde{x}, p)$ as the lub of S. As $(\tilde{X}, \tilde{x}, p)$ is the lub of the set S, we have $(\tilde{X}', \tilde{x}', p') \geq (\tilde{X}, \tilde{x}, p)$ for all $(\tilde{X}', \tilde{x}', p') \in S$. Now as f is an isotone function, so we have $(\tilde{X}', \tilde{x}', p') \geq f(\tilde{X}', \tilde{x}', p') \geq f(\tilde{X}, \tilde{x}, p)$ for all $(\tilde{X}', \tilde{x}', p') \in S$. Hence $(\tilde{X}, \tilde{x}, p) = lub\ S \geq f(\tilde{X}, \tilde{x}, p)$. Again as f is isotone function, it follows that $f(\tilde{X}, \tilde{x}, p) \geq f(f(\tilde{X}, \tilde{x}, p))$, whence $f(\tilde{X}, \tilde{x}, p) \in S$. Now since $(\tilde{X}, \tilde{x}, p) = lub\ S$, it follows that $f(\tilde{X}, \tilde{x}, p) \geq (\tilde{X}, \tilde{x}, p)$.
Hence $f((\tilde{X}, \tilde{x},, p)) = (\tilde{X}, \tilde{x}, p)$.

Example 3.2: Let $T = S^1 \times S^1 = Torus$. Then $\pi(T) = \mathbf{Z} \oplus \mathbf{Z}$, which is abelian. Also $\mathbb{R}^2$ is an universal covering space of T .Hence, by **Theorem 1.5**, $(C(T), \geq)$ is a lattice, by **Theorem 2.2**,$(C(T), \geq)$ is a modular lattice. By **Theorem 2.3** it is a bounded lattice and by **Theorem 2.4** , it is a complete lattice.

Example 3.3: Let $\mathbb{R}\mathbf{P}^n$ be the real projective n-space. $\pi(\mathbb{R}\mathbf{P}^n) = C_2$, which is a cyclic group of order 2, hence an abelian group. Again S^n is an universal covering space of $\mathbb{R}\mathbf{P}^n$ for $n \geq 2$. Hence, by **theorem 1.5**, $(C(\mathbb{R}\mathbf{P}^n), \geq)$ is a lattice for $n \geq 2$, by **theorem 2.2**,$(C(\mathbb{R}\mathbf{P}^n), \geq)$ is a modular lattice for $n \geq 2$. By **theorem 2.3** it is a bounded lattice for $n \geq 2$ and by **theorem 2.4** ,it is a complete lattice for $n \geq 2$.

ACKNOWLEDGEMENT
Second author is thankful to the UGC for grant JRF(201920-19J6069257).The third author is thankful to the CSIR, HRDG, for grant JRF(File No.:08/525 (0006)/2019-EMR-I)during the tenure of which this work was done.

CONFLICT OF INTEREST STATEMENT:
This paper doesn't any conflict of interest statement as it is produced and processed under purely private interest.

REFERENCES

[1] W.S.Massey,Algebraic topology an Introduction, Harecourt ,Brace & Word,Inc.,1967.
[2] E.H.spanier, Algebraic Topology, Tata McGraw-Hill Publishing Com- pany Ltd,New Delhi,1978.
[3] J.R.Munkres,Topology: A first course, Prentice-Hall, Englewood Cliffs, N.J.,1975.
[4] Birkhoff, G., Lattice theory, Amer. Math. Soc., Providence, R.I., 1967.
[5] Adhikari, M.R.; Rana,P.K. and Adhikari,A.(1999): A study of covering spaces through a functor, Review Bull Cal.Math.Soc.,7,(1 and 2),75-80.
[6] Adhikari,M.R.(2016):Basic Algebraic Topology and its Applications, Springer.