GENERALIZED HYERS-ULAM - RASSISA STABILITY OF AN ADDITIVE \((\beta_1; \beta_2)\)-FUNCTIONAL INEQUALITIES WITH THREE VARIABLES IN COMPLEX BANACH SPACE

LY VAN AN
Faculty of Mathematics Teacher Education, Tay Ninh University, Vietnam.

Email: lyvanan145@gmail.com (lyvananvietnam@gmail.com).

Abstract. In this paper we study to solve the of additive \((\beta_1; \beta_2)\)-functional inequality with three-variables and their Hyers-Ulam stability. First are investigated complex Banach spaces with a fixed point method and last are investigated in complex Banach spaces with a direct method. Then Hyers-Ulam stability of these equation are given and proven. These are the main results of this paper.

Keywords: Additive \((\beta_1; \beta_2)\)-Functional inequality; Fixed Point Method; Direct Method; Banach Space; Hyers-Ulam Stability.

Mathematics Subject Classification: 46S10; 39B62; 39B52; 47H10.

I. INTRODUCTION

Let \(X\) and \(Y\) be a normed spaces on the same field \(K\), and \(f : X \to Y\). We use the notation \(\cdot\) for all the norm on both \(X\) and \(Y\). In this paper, we investigate additive \((\beta_1; \beta_2)\)-functional inequality when \(X\) be a real or complex normed space and \(Y\) a complex Banach spaces. We solve and prove the Hyers-Ulam stability of the following additive \((\beta_1; \beta_2)\)-functional inequality

\[
\left\| 2f\left(\frac{x + y}{2} + \frac{z}{4}\right) - f(x) - f\left(y + \frac{z}{2}\right) \right\|_Y \\
\leq \left\| \beta_1 \left(f\left(x + y + \frac{z}{2}\right) + f\left(x - y - \frac{z}{2}\right) - 2f(x)\right) \right\|_Y \\
+ \left\| \beta_2 \left(f\left(x + y + \frac{z}{2}\right) - f(x) - f\left(y + \frac{z}{2}\right)\right) \right\|_Y
\]

We solve and prove the Hyers-Ulam stability of the following additive \(|\beta_1| + |\beta_2| < 1\).

Note that in the preliminaries we just recap some of the most essential properties for the above problem and for the specific problem, please see the document. The Hyers-Ulam stability was first investigated for functional equation of Ulam in [28] concerning the stability of group homomorphisms.

The functional equation

\(f(x + y) = f(x) + f(y)\)
is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. The Hyers [13] gave first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers’ Theorem was generalized by Aoki [1] additive mappings and by Rassias [26] for linear mappings considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach. The stability of quadratic functional equation was proved by Skof [27] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Park [24], [25] defined additive $\gamma$-functional inequalities and proved the Hyers-Ulam stability of the additive $\gamma$-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors on the world. We recall a fundamental result in fixed point theory. Recently, in [3], [4], [21], [22], [24], [25] the authors studied the Hyers-Ulam stability for the following functional inequalities:

$$\begin{align*}
\left\| f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x+y}{2}\right) - f(z) \right\| &\leq \left\| f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - \frac{1}{2} f\left(\frac{x+y}{2}\right) - \frac{1}{2} f(z) \right\| \tag{1.2}
\left\| f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - \frac{1}{2} f\left(\frac{x+y}{2}\right) - \frac{1}{2} f(z) \right\| &\leq \left\| f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f(z) \right\| \tag{1.3}
\left\| f(x+y) - f(x) - f(y) \right\| &\leq \beta\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{1.4}
\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| &\leq \beta\left(f(x+y) - f(x) - f(y)\right) \tag{1.5}
\end{align*}$$

and

$$\begin{align*}
\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2f(z) \right\| &\leq \beta\left(2f\left(\frac{x+y}{2} + \frac{z}{2}\right) + 2f\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f(z) \right) \tag{1.6}
\left\| 2f\left(\frac{x+y}{2} + \frac{z}{2}\right) + 2f\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f(z) \right\| &\leq \beta\left(f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2f(z) \right) \tag{1.7}
\end{align*}$$

finally

$$\begin{align*}
\left\| f(x+y) - f(x) - f(y) \right\| &\leq \beta_1\left(f(x+y) + f(x-y) - 2f(x)\right) + \beta_2\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{1.8}
\end{align*}$$
in complex Banach spaces
In this paper, we solve and proved the Hyers-Ulam stability for \((\beta_1; \beta_2)\)-functional inequalities (1.1), ie the \((\beta_1; \beta_2)\)-functional inequalities with three variables. Under suitable assumptions on spaces \(X\) and \(Y\), we will prove that the mappings satisfying the \((\beta_1; \beta_2)\)-functional inequatilies (1.1). Thus, the results in this paper are generalization of those in [3],[4],[14],[21] for \((\beta_1; \beta_2)\)-functional inequalities with three variables.

The paper is organized as follows: In section preliminarier we remind some basic notations in [3,7] such as complete generalized metric space and Solutions of the inequalities.

Section 3: In this section, I use the method of the fixed to prove the Hyers-Ulam stability of the additive \((\beta_1; \beta_2)\)- functional inequalities (1.1) when \(X\) be a real or complete normed space and \(Y\) complex Banach space.

Section 4: In this section, I use the method of directly determining the solution for (1.1) when \(X\) be a real or complete normed space and \(Y\) complex Banach space.

2. Preliminaries


Theorem 2.1. Let \(X; d\) be a complete generalized metric space and let \(J : X \rightarrow X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then for each given element \(x \in X\), either

\[d(J^n; J^{n+1})= 1\]

for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that

1. \(d(J^n, J^{n+1}) < \infty, \forall n \geq n_0\);
2. The sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J\);
3. \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X | d(J^n, J^{n+1}) < \infty\}\);
4. \(d(y, y^*) \leq \frac{1}{1-L}d(y, Jy) \forall y \in Y\)

2.2. Solutions of the inequalities. The functional equation

\[f(x + y) = f(x) + f(y)\]

is called the Cauchuy equation. In particular, every solution of the cauchuy equation is said to be an additive mapping.

3. Establish The Solution of The Additive \((\beta_1; \beta_2)\)-Function Inequalities Using a Fixed Point Method

Now, we first study the solutions of (1.1). Note that for these inequalities, when \(X\) be a real or complete normed space and \(Y\) complex Banach space
Lemma 3.1. A mapping $f : X \to Y$ satisfies $f(0) = 0$ and

$$\left\| 2f \left( \frac{x + y}{2} + \frac{z}{4} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right\|_Y$$

$$\leq \left\| \beta_1 \left( f \left( x + y + \frac{z}{2} \right) + f \left( x - y - \frac{z}{2} \right) - 2f(x) \right) \right\|_Y$$

$$+ \left\| \beta_2 \left( f \left( x + y + \frac{z}{2} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right) \right\|_Y$$

(3.1)

for all $x, y, z \in X$ if and only if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1)

Letting $y = z = 0$ in (3.1), we get

$$\left\| 2f \left( \frac{x}{2} \right) - f(x) \right\|_Y \leq 0$$

Thus

$$f \left( \frac{x}{2} \right) = \frac{1}{2} f(x)$$

(3.2)

for all $x \in X$ It follows from (3.1) and (3.2) that

$$\left\| f \left( x + y + \frac{z}{2} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right\|_Y$$

$$= \left\| 2f \left( \frac{x + y}{2} + \frac{z}{4} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right\|_Y$$

$$\leq \left\| \beta_1 \left( f \left( x + y + \frac{z}{2} \right) + f \left( x - y - \frac{z}{2} \right) - 2f(x) \right) \right\|_Y$$

$$+ \left\| \beta_2 \left( f \left( x + y + \frac{z}{2} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right) \right\|_Y$$

(3.3)

and so

$$\left( 1 - \beta_2 \right) \left\| f \left( x + y + \frac{z}{2} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right\|_Y$$

$$\leq \beta_1 \left\| f \left( x + y + \frac{z}{2} \right) + f \left( x - y - \frac{z}{2} \right) - 2f(x) \right\|_Y$$

(3.4)

Letting $u = x + y + \frac{z}{2}, v = x - y - \frac{z}{2}$ in (3.4), we get
\[
\left(1 - |\beta_2|\right) \left\| f(u) - f\left(\frac{u + v}{2}\right) - f\left(\frac{u - v}{2}\right) \right\|_Y \leq |\beta_1| \left\| f(u) + f(v) - 2f\left(\frac{u + v}{2}\right) \right\|_Y
\]
\[
(3.5)
\]

for all \(u, v \in X\)

and so

\[
\frac{1}{2} \left(1 - |\beta_2|\right) \left\| f(u + v) + f(u - v) - 2f(u) \right\|_Y \leq |\beta_1| \left\| f(u + v) - f(u) - f(v) \right\|_Y \]
\[
(3.6)
\]

for all \(u, v \in X\). It follows from (3.4) and (3.5) that

\[
\frac{1}{2} \left(1 - |\beta_2|\right)^2 \left\| f\left(x + y + \frac{z}{2}\right) - f(x) + f\left(y + \frac{z}{2}\right) \right\|_Y \leq |\beta_1|^2 \left\| f\left(x + y + \frac{z}{2}\right) - f(x) - f\left(y + \frac{z}{2}\right) \right\|_Y
\]
\[
(3.7)
\]

Since \(\sqrt{2}|\beta_1| + |\beta_2| < 1\)

and so

\[
f\left(x + y + \frac{z}{2}\right) - f(x) - f\left(y + \frac{z}{2}\right) = 0
\]

for all \(x, y, z \in X\). Thus \(f\) is additive. \(\square\)

**Theorem 3.2.** Suppose \(\varphi : X^3 \rightarrow [0, \infty)\) be a function such that there exists an \(L < 1\) with

\[
\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2} \varphi(x, y, z)
\]
\[
(3.8)
\]

for all \(x, y, z \in X\). Give \(f : X \rightarrow Y\) be a mapping satisfy \(f(0) = 0\) and

\[
\left\| 2f\left(\frac{x + y}{2} + \frac{z}{4}\right) - f(x) - f\left(y + \frac{z}{2}\right) \right\|_Y \leq \left\| \beta_1\left(f\left(x + y + \frac{z}{2}\right) + f\left(x - y - \frac{z}{2}\right) - 2f(x)\right) \right\|_Y
\]

\[
+ \left\| \beta_2\left(f\left(x + y + \frac{z}{2}\right) - f(x) - f\left(y + \frac{z}{2}\right)\right) \right\|_Y + \varphi(x, y, z)
\]
\[
(3.9)
\]

for all \(x, y, z \in X\).

Then there exists a unique mapping \(H : X \rightarrow Y\) such that

\[
\left\| f(x) - H(x) \right\|_Y \leq \frac{1}{1 - L} \varphi(x, 0, 0)
\]
\[
(3.10)
\]

for all \(x \in X\).
Proof. Letting \( y = z = 0 \) in (3.9), we get
\[
\left\| 2f \left( \frac{x}{2} \right) - f(x) \right\|_{\mathcal{Y}} \leq \varphi(x, 0, 0)
\]  
(3.11)

for all \( x \in \mathbb{X} \).

Consider the set
\[
\mathcal{S} := \left\{ h : \mathbb{X} \to \mathcal{Y}, h(0) = 0 \right\}
\]

and introduce the generalized metric on \( \mathcal{S} \):
\[
d(g, h) := \inf \left\{ \lambda \in \mathbb{R}_+ : \left\| g(x) - h(x) \right\| \leq \lambda \varphi(x, 0, 0), \forall x \in \mathbb{X} \right\},
\]
where, as usual, \( \inf \phi = +\infty \). It easy to show that \( (\mathcal{S}, d) \) is complete (see[16]). Now we consider the linear mapping \( J : \mathcal{S} \to \mathcal{S} \) such that
\[
Jg(x) := 2g \left( \frac{x}{2} \right)
\]
for all \( x \in \mathbb{X} \). Let \( g, h \in \mathcal{S} \) be given such that \( d(g, h) = \epsilon \) then
\[
\left\| g(x) - h(x) \right\| \leq \epsilon \varphi(x, 0, 0)
\]
for all \( x \in \mathbb{X} \). Hence
\[
\left\| Jg(x) - Jh(x) \right\| = \left\| 2g \left( \frac{x}{2} \right) - 2h \left( \frac{x}{2} \right) \right\| \leq 2\epsilon \varphi \left( \frac{x}{2}, 0, 0 \right)
\]
\[
\leq 2\epsilon \frac{L}{2} \varphi(x, 0, 0) \leq L\epsilon \varphi(x, 0, 0)
\]
for all \( x \in \mathbb{X} \). So \( d(g, h) = \epsilon \) implies that \( d(Jg, Jh) \leq L\epsilon \). This means that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for all \( g, h \in \mathbb{X} \). It follows from (3) that \( d(f, Jf) \leq 1 \). Follow Theorem 1.1, there exists a mapping \( H : \mathbb{X} \to \mathcal{Y} \) satifying the following:

(1) \( H \) is a fixed point of \( J \), i.e.,
\[
H(x) = 2H \left( \frac{x}{2} \right)
\]
(3.12)
for all \( x \in \mathbb{X} \). The mapping \( H \) is a unique fixed point \( J \) in the set
\[
\mathbb{M} = \left\{ g \in \mathcal{S} : d(f, g) < \infty \right\}
\]
This implies that \( H \) is a unique mapping satisfying (3.12) such that there exists a \( \lambda \in \left( 0, \infty \right) \) satisfying
\[
\left\| f(x) - H(x) \right\| \leq \lambda \varphi(x, 0, 0)
\]
for all \( x \in \mathbb{X} \).
(2) \( d\left(J^l f, H\right) \to 0 \) as \( l \to \infty \). This implies equality

\[
\lim_{l \to \infty} 2^n f\left(\frac{x}{2^n}\right) = H(x)
\]

for all \( x \in X \).

(3) \( d\left(f, H\right) \leq \frac{1}{1-L} d\left(f, J^l f\right) \), which implies

\[
\left\|f(x) - H(x)\right\| \leq \frac{L}{1-L} \varphi(x, 0, 0)
\]

for all \( x \in X \). It follows (3.8) and (3.9) that

\[
\left\|2H\left(\frac{x+y}{2} + \frac{z}{4}\right) - H(x) - H\left(y + \frac{z}{2}\right)\right\|_Y
\]

\[
= \lim_{n \to \infty} 2^n \left\|2f\left(\frac{x+y}{2^n} + \frac{z}{2^{n+1}}\right) - f\left(\frac{x}{2^n} + \frac{z}{2^{n+1}}\right)\right\|_Y
\]

\[
\leq \lim_{n \to \infty} 2^n \beta_1 \left\|f\left(\frac{x+y}{2^n} + \frac{z}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n} - \frac{z}{2^{n+1}}\right) - 2f\left(\frac{x}{2^n}\right)\right\|_Y
\]

\[
+ \lim_{n \to \infty} 2^n \beta_2 \left\|f\left(\frac{x+y}{2^n} + \frac{z}{2^{n+1}}\right) + f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n} + \frac{z}{2^{n+1}}\right)\right\|_Y + \lim_{n \to \infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)
\]

\[
= \|\beta_1 (H(x+y+z)+H(x+y-z/2)+2H(x))\|_Y + \|\beta_2 (H(x+y+z)/2) - H(x) - H(y+z/2)\|_Y
\]

for all \( x, y, z \in X \). So

\[
\left\|2H\left(\frac{x+y}{2} + \frac{z}{4}\right) - H(x) - H\left(y + \frac{z}{2}\right)\right\|_Y
\]

\[
\leq \|\beta_1 (H(x+y+z)+H(x+y-z)+2H(x))\|_Y
\]

\[
+ \|\beta_2 (H(x+y+z)/2) - H(x) - H(y+z/2)\|_Y
\]

for all \( x, y, z \in X \). By Lemma 3.1, the mapping \( H : X \to Y \) is additive.

\[\square\]

**Theorem 3.3.** Let \( \varphi : X^3 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[
\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)
\]

(3.13)
for all \(x, y, z \in \mathbb{X}\). Let \(f : \mathbb{X} \rightarrow \mathbb{Y}\) be a mapping satisfy \(f(0) = 0\) and

\[
\left\| 2f\left(\frac{x + y + z}{4}\right) - f(x) - f\left(y + \frac{z}{2}\right) \right\|_Y \\
\leq \left\| \beta_1 \left( f(x + y + \frac{z}{2}) + f\left(x - y - \frac{z}{2}\right) - 2f(x) \right) \right\|_Y \\
+ \left\| \beta_2 \left( f(x + y + \frac{z}{2}) - f(x) - f\left(y + \frac{z}{2}\right) \right) \right\|_Y + \varphi(x, y, z) \tag{3.14}
\]

for all \(x, y, z \in \mathbb{X}\).

Then there exists a unique mapping \(H : \mathbb{X} \rightarrow \mathbb{Y}\) such that

\[
\left\| f(x) - H(x) \right\|_Y \leq \frac{L}{1 - L} \varphi(x, 0, 0) \tag{3.15}
\]

for all \(x \in \mathbb{X}\).

Proof. Letting \(y = z = 0\) in (3.14), we get

\[
\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\|_Y \leq \varphi(x, 0, 0) \tag{3.16}
\]

and so

\[
\left\| f(x) - \frac{1}{2}f\left(2x\right) \right\|_Y \leq \frac{1}{2} \varphi\left(2x, 0, 0\right) \leq L \varphi(x, 0, 0) \tag{3.17}
\]

for all \(x \in \mathbb{X}\).

Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 3.2

Now we consider the mapping \(J : S \rightarrow S\) such that

\[
Jg(x) := \frac{1}{2}g(2x)
\]

for all \(x \in X\).

It follows (3.17) that \(d\left(f, Jf\right) \leq \frac{1}{2} \varphi(2x, 0, 0) \leq L \varphi(x, 0, 0)\). So

for all \(x \in X\).

The rest of the proof is similar to proof of Theorem 3.2. \(\square\)

From proving the theorems we have consequences:

**Corollary 3.4.** Let \(r > 1\) and \(\theta\) be nonnegative real numbers and let \(f : \mathbb{X} \rightarrow \mathbb{Y}\) be a mapping satisfy \(f(0) = 0\) and
4. Establish The Solution of The Additive ($\beta_1; \beta_2$)-Function Inequalities Using A Aireect Method

Next, we study the solutions of (1.1). Note that for these inequalities, when $X$ be a real or complete normed space and $Y$ complex Banach space.

**Theorem 4.1.** Let $\varphi : X^3 \to [0, \infty)$ be a function such that

$$
\psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty
$$

and let $f : X \to Y$ be a mapping $f(0) = 0$ and
\[ \left\| 2f \left( \frac{x+y}{2} + \frac{z}{4} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right\| \leq \left\| \beta_1 \left( f \left( x + y + \frac{z}{2} \right) + f \left( x - y - \frac{z}{2} \right) - 2f(x) \right) \right\| + \phi(x, y, z) \quad (4.2) \]

for all \( x, y, z \in \mathbb{X} \).

Then there exists a unique additive mapping \( H : \mathbb{X} \to \mathbb{Y} \) such that

\[ \left\| f(x) - H(x) \right\| \leq \phi(x, 0, 0) \quad (4.3) \]

for all \( x \in \mathbb{X} \).

**Proof.** Let \( y = z = 0 \) in (4.9), we get

\[ \left\| 2f \left( \frac{x}{2} \right) - f(x) \right\| \leq \phi(x, 0, 0) \quad (4.4) \]

for all \( x \in \mathbb{X} \). So

\[ \left\| f(x) - 2f \left( \frac{x}{2} \right) \right\| \leq \phi(x, 0, 0) \quad (4.5) \]

for all \( x \in \mathbb{X} \). Hence

\[ \left\| 2^l f \left( \frac{x}{2^l} \right) - 2^m f \left( \frac{x}{2^m} \right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j f \left( \frac{x}{2^j} \right) - 2^{j+1} f \left( \frac{x}{2^{j+1}} \right) \right\| \leq \sum_{j=l}^{m-1} 2^j \phi \left( \frac{x}{2^j}, 0, 0 \right) \quad (4.6) \]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in \mathbb{X} \). It follows from (4.6) that the sequence \( \left\{ 2^n f \left( \frac{x}{2^n} \right) \right\} \) is a Cauchy sequence for all \( x \in \mathbb{X} \). Since \( \mathbb{Y} \) is complete, the sequence \( \left\{ 2^n f \left( \frac{x}{2^n} \right) \right\} \) converges. So one can define the mapping \( H : \mathbb{X} \to \mathbb{Y} \) by

\[ H(x) := \lim_{n \to \infty} \frac{1}{2^n} f \left( 2^n x \right) \quad (4.7) \]

It follows from (4.1) and (4.9) that
**Theorem 4.2.** Let \( \varphi : \mathbb{X}^+ \to [0, \infty) \) be a function such that
\[
\psi(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{2^{2j}} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]  
and let \( f : \mathbb{X} \to \mathbb{Y} \) be a mapping \( f(0) = 0 \) and
\[
\left\| 2f\left(\frac{x+y+z}{2}\right) - f(x) - f\left(y+\frac{z}{2}\right) \right\|_\mathbb{Y}
\leq \left\| \beta_1 \left(f\left(x+y+\frac{z}{2}\right) + f\left(x-y-\frac{z}{2}\right) - 2f(x)\right) \right\|_\mathbb{Y}
\]
\[
+ \left\| \beta_2 \left(f\left(x+y+\frac{z}{2}\right) - f(x) - f\left(y+\frac{z}{2}\right) \right) \right\|_\mathbb{Y} + \varphi(x, y, z)
\]  
for all \( x, y, z \in \mathbb{X} \).

Then there exists a unique additive mapping \( H : \mathbb{X} \to \mathbb{Y} \) such that
\[
\left\| f(x) - H(x) \right\|_\mathbb{Y} \leq \psi(x, 0, 0)
\]  
for all \( x \in \mathbb{X} \).

**Proof.** Let \( y = z = 0 \) in (4.11), we get
\[
\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\|_\mathbb{Y} \leq \varphi(x, 0, 0)
\]  
for all \( x \in \mathbb{X} \). So
\[
\left\| f(x) - \frac{1}{2} f\left(2^0 x\right) \right\|_\mathbb{Y} \leq \frac{1}{2} \varphi(2x, 0, 0)
\]  
for all \( x \in \mathbb{X} \). Hence
\[
\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\|_\mathbb{Y}
\leq \sum_{j=l}^{m-1} \left\| 2^j \frac{1}{2^j} f\left(2^j x\right) - 2^{j+1} \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\|_\mathbb{Y}
\]
\[
\leq \sum_{j=l}^{m-1} \frac{1}{2^{2j+1}} \varphi\left(2^j x, 0, 0\right)
\]  
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in \mathbb{X} \). It follows from (4.15) that the sequence \( \left\{ \frac{1}{2^n} f\left(2^n x\right) \right\} \) is a Cauchy sequence for all \( x \in \mathbb{X} \). Since \( \mathbb{Y} \) is complete, the sequence \( \left\{ \frac{1}{2^n} f\left(2^n x\right) \right\} \) converges. So one can define the mapping \( H : \mathbb{X} \to \mathbb{Y} \) by
\[
H(x) := \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right)
\]  
x \in \mathbb{X} \). The rest of the proof is similar to the proof of Theorem 4.1. \( \square \)
\begin{align*}
&\left\|2H\left(\frac{x+y}{2} + \frac{z}{4}\right) - H(x) - H\left(y + \frac{z}{2}\right)\right\|_Y \\
= &\quad \lim_{n \to \infty} 2^n \left\|2f\left(\frac{x+y}{2^{n+1}} + \frac{z}{2^{2n+2}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n} + \frac{z}{2^{n+1}}\right)\right\|_Y \\
\leq &\quad \lim_{n \to \infty} 2^n \beta_1 \left\|f\left(\frac{x+y}{2^n} + \frac{z}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n} - \frac{z}{2^{n+1}}\right) - 2f\left(\frac{x}{2^n}\right)\right\|_Y \\
+ &\quad \lim_{n \to \infty} 2^n \beta_2 \left\|f\left(\frac{x+y}{2^n} + \frac{z}{2^{n+1}}\right) + f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n} + \frac{z}{2^{n+1}}\right)\right\|_Y + \lim_{n \to \infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
= &\quad \beta_1 \left(H\left(x+y + \frac{z}{2}\right) + H\left(x-y - \frac{z}{2}\right) - 2H(x)\right)\right\|_Y \\
+ &\quad \beta_2 \left(H\left(x+y + \frac{z}{2}\right) - H(x) - H\left(y + \frac{z}{2}\right)\right)\right\|_Y \tag{4.8}
\end{align*}

for all $x, y, z \in \mathbb{X}$. So

\begin{align*}
&\left\|2H\left(\frac{x+y}{2} + \frac{z}{4}\right) - H(x) - H\left(y + \frac{z}{2}\right)\right\|_Y \\
\leq &\quad \beta_1 \left(H\left(x+y + \frac{z}{2}\right) + H\left(x-y - \frac{z}{2}\right) - 2H(x)\right)\right\|_Y \\
+ &\quad \beta_2 \left(H\left(x+y + \frac{z}{2}\right) - H(x) - H\left(y + \frac{z}{2}\right)\right)\right\|_Y \tag{4.9}
\end{align*}

\begin{align*}
H\left(x+y + \frac{z}{2}\right) - H(x) - H\left(y + \frac{z}{2}\right) = 0
\end{align*}

for all $x, y, z \in \mathbb{X}$. By Lemma 3.1, the mapping $H : \mathbb{X} \to \mathbb{Y}$ is additive.

Now, let $T : \mathbb{X} \to \mathbb{Y}$ be another additive mapping satisfying (4.3). Then we have

\begin{align*}
&\left\|H(x) - T(x)\right\|_Y = \left\|2^q H\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\|_Y \\
\leq &\quad 2^q H\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|_Y + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|_Y \\
\leq &\quad 2^{q+1} \psi\left(\frac{x}{2^q}, 0, 0\right)
\end{align*}

which tends to zero as $q \to \infty$ for all $x \in \mathbb{X}$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbb{X}$. This proves the uniqueness of $H$. \qed
Corollary 4.3. Let $r > 1$ and $\theta$ be nonnegative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping satisfy $f(0) = 0$ and

$$
\left\| 2f \left( \frac{x+y}{2} + \frac{z}{4} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right\|_Y
\leq \left\| \beta_1 \left( f \left( x + y + \frac{z}{2} \right) + f \left( x - y - \frac{z}{2} \right) - 2f(x) \right) \right\|_Y
+ \left\| \beta_2 \left( f \left( x + y + \frac{z}{2} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right) \right\|_Y + \theta \left( \|x\|^r + \|y\|^r + \|z\|^r \right)
$$

(4.17)

for all $x, y, z \in \mathbb{X}$. Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$
\left\| f(x) - H(x) \right\|_Y \leq \frac{2r\theta}{2^r - 2} \|x\|^r
$$

(4.18)

for all $x \in \mathbb{X}$.

Corollary 4.4. Let $r < 1$ and $\theta$ be nonnegative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping satisfy $f(0) = 0$ and

$$
\left\| 2f \left( \frac{x+y}{2} + \frac{z}{4} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right\|_Y
\leq \left\| \beta_1 \left( f \left( x + y + \frac{z}{2} \right) + f \left( x - y - \frac{z}{2} \right) - 2f(x) \right) \right\|_Y
+ \left\| \beta_2 \left( f \left( x + y + \frac{z}{2} \right) - f(x) - f \left( y + \frac{z}{2} \right) \right) \right\|_Y + \theta \left( \|x\|^r + \|y\|^r + \|z\|^r \right)
$$

(4.19)

for all $x, y, z \in \mathbb{X}$. Then there exists a unique mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$
\left\| f(x) - H(x) \right\|_Y \leq \frac{2r\theta}{2^r - 2} \|x\|^r
$$

(4.20)

for all $x \in \mathbb{X}$.

5. Conclusion

In this paper, I have shown that the solutions of the $\beta_1; \beta_2$-functional inequalities are additive mappings. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper, which are the generalization of the results [3,4,14,21].

6. References


[21] Choonekil Park. the stability an of additive $(p_1,p_2)$-functional inequality in Banach space Journal of mathematical Inequalities Volume 13, Number 1 (2019), 95-104.


