A LINEAR RELATION BETWEEN CURVATURES AND FUNDAMENTAL FORMS

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Abstract

In this work, a linear relation between high order fundamental forms and curvatures was obtained by using Cayley-Hamilton theorem.

1 INTRODUCTION

Definition 1.1 Let M be a hypersurface of E^n and N be a unit normal field of M. Then the mapping S on M defined by

$$S(X) = D_X N \text{ for all } X \in X(M)$$

is called a shape operator or Weingarten Mapping of M, where D is the Riemann connexion on E^n (Hicks,1974, pp:21).

Definition 1.2 Let M be a hypersurface of E^n .The fundamental forms on M can now be defined in terms of S and the inner product. If X and Y are in $X(M)$, then

$$I(X,Y) = < X, Y >,$$

$$I^2 = II(X,Y) = < S(X), Y >,$$

$$I^3 = III(X,Y) = < S^2(X), Y >,$$

$$I^4 = IV(X,Y) = < S^3(X), Y >,$$

etc, and these forms are called the first, second, third,etc, fundamental forms on M (Hicks,1974).

Definition 1.3 (Cayley- Hamilton Theorem) Consider the n-square matrix $\mathbf{A} = [\mathbf{a_{ij}}]$ having characteristic matrix $\mathbf{A} - \lambda\mathbf{I}$ and characteristic equation $P(\lambda) = det(\mathbf{A} - \lambda\mathbf{I}) = 0$ (Ayres,1974).

Definition 1.4 Let M be a hypersurface in E^{n+1} and $T_M(P)$ be a tangent space on M, at $P \in M$. If S_P denotes the shape operator on M, at $P \in M$, then

$$S_P : T_M(P) \longrightarrow T_M(P)$$

is a linear mapping. If we denote the characteristic vectors by $x_1, x_2, ..., x_n$ of S_P then $\lambda_1, \lambda_2, ..., \lambda_n$ are the principle curvatures and $x_1, x_2, ..., x_n$ are the principle directions of M, at $P \in M$. On the other hand, if we use the notions

$K_1(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i=1}^{n} \lambda_i$

$K_2(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i<j}^{n} \lambda_i \lambda_j$

$K_3(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i<j<k}^{n} \lambda_i \lambda_j \lambda_k$

$\vdots$

$K_n(\lambda_1, \lambda_2, ..., \lambda_n) = \prod_{i=1}^{n} \lambda_i$

then the characteristic polinomial of $S(P)$ becomes

$P_{S(P)}(\lambda) = \lambda^n + (-1)K_1\lambda^{n-1} + ... + (-1)^n K_n$

and $K_i, 1 \leq i \leq n$ are uniquely determined, where the functions K_i are called the higher ordered Gaussian curvatures of the hypersurface M (Özdamar-Hacısalihoğlu, 1977, Kobayashi-Nomizo, 1969).

Theorem 1.1 Let M be a hypersurface of E^3. Then the following relation holds between the first, second and third fundamental forms of M:

$$III - HII + KI = 0$$

where H and K denote the mean curvature and Gaussian curvature of M (Hacısalihağlu, 2003).

2 GENERALIZED THEOREM

Theorem 2.1 Let M be a hypersurface of E^{2n+1}. The fundamental forms $I, I^2, ..., I^{2n+1}$ on M and higher ordered Gaussian curvatures $K_1, K_2, ..., K_{2n}$ then

$$\sum_{i=0}^{2n} K_i I^{2n+1} \equiv 0 \text{ and } K_0 = 1$$

Proof: M be a hypersurface of E^{2n+1}, $dimM = 2n$, then $dimT_M(P) = 2n$. Therefore $S : T_M(P) \longrightarrow T_M(P)$ the characteristic polinomial of shape operator is $2n$ order. Moreover, since $K_1, K_2, ..., K_{2n}$ curvatures are zeros of this polinomial of S is

$$P_S(\lambda) = \lambda^{2n} + (-1)K_1\lambda^{2n-1} + ... + (-1)^{2n}K_{2n}$$

According to Cayley-Hamilton theorem , S is the zero of this polinomial. Then

$$S^{2n} + (-1)K_1S^{2n-1} + ... + (-1)^{2n}K_{2n}I_{2n} = 0$$

$$(S^{2n} + (-1)K_1S^{2n-1} + ... + (-1)^{2n}K_{2n}I_{2n})(X_P) = 0 \ \forall \ \mathrm{X}_P \in T_M(P)$$

$$\left\langle (S^{2n} + (-1)K_1S^{2n-1} + ... + (-1)^{2n}K_{2n}I_{2n})(X_P), Y_P \right\rangle = 0 \ \forall \mathrm{X}_P \in T_M(P)$$

$$\left\langle (S^{2n}(X_P), Y_P \right\rangle + (-1)K_1 \left\langle S^{2n-1}(X_P), Y_P \right\rangle + ... + (-1)^{2n}K_{2n} \langle (X_P), Y_P \rangle = 0$$
$$\mathrm{I}^{2n+1}(X_P, Y_P) - K_1I^{2n}(X_P, Y_P) + ... + K_{2n}I(X_P, Y_P) = 0$$

$$I^{2n+1} - K_1I^{2n} + ... + K_{2n}I = 0$$

Therefore

$$\sum_{i=0}^{2n}(-1)^{2+i}K_iI^{2n+1-i} = 0, \ K_0 = 1$$

REMARK: If special case $n = 1$, a linear relation between fundamental form and curvatures, well-known in E^3 is obtained $III - HII + KI = 0$ Let M be a hypersurface of E^{2n+1}.

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