Note on Same Amazing properties of Perfect Numbers

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Abstract. Number theory has been a fascinating topic for many mathematicians because of the relationships they can find between numbers and their divisors. One of which are, perfect numbers. They date back to ancient Greek mathematics when they studied the relationships between numbers and their divisors. As of 2018, there are currently 50 perfect numbers that have been found. The first four are 6, 28, 496, and 8128.The largest containing more than 23 million digits. Perfect numbers are still being found to this day and no one has found an odd perfect number yet. Euclid proved a formula for finding perfect numbers: If $2^n - 1$ is prime, then $[2^{(n-1)}](2^n - 1)$ is perfect. In other words, n is a number whose positive divisors sums to n.

Key Words: Prime Numbers and Perfect numbers.

1. Introduction and Background

Throughout history, there have been studies on perfect numbers. It is not known when perfect numbers were first studied and indeed the first studies may go back to the earliest times when numbers first aroused curiosity [6]. It is rather likely, although not completely certain, that the Egyptians would have come across such numbers naturally given the way their methods of calculation worked, where detailed justification for this idea is given [6]. Perfect numbers were studied by Pythagoras and his followers, more for their mystical properties than for their number theoretic properties [6]. Although, the four perfect numbers 6, 28, 496 and 8128 seem to have been known from ancient times and there is no record of these discoveries [6]. The First recorded mathematical result concerning perfect numbers which is known occurs in Euclid's Elements written around 300BC [6].

Proof

If $2^k - 1$ ($k>1$) is prime, then $n = 2^{k-1}(2^k - 1)$ is a perfect number.

Proof: We will show that n = sum of its proper factors. We will find all the proper factors of $2^{k-1}(2^k - 1)$, and add them. Since 2^k-1 is prime, let $p = 2^k-1$. Then n = $p(2^k-1)$

Let us list all factors of 2^{k-1} and other proper factors of n as follows.

Factors of $2k$-1	Other Proper Factors
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-2}p$

Adding the first column, we get:

$$1+2+2^2+2^3 \ldots + 2^{k-3}+2^{k-2}+2^{k-1}$$
$$=2^k-1$$
$$=p$$

Adding the second column, we get:

$$p+2p+2^2p+2^3p\ldots+2^{k-4}p+2^{k-3}p+2^{k-2}p$$
$$=p\left(1+2+2^2+\ldots+2^{k-2}\right)$$
$$=\left(2^{k-1}-1\right)p$$

Now adding the two columns together, we get:

$$p+p\left(2^{k-1}-1\right)$$
$$=p\left(1+2^{k-1}-1\right)$$
$$=p\left(2^{k-1}\right)$$
$$=n$$

Hence. n is a perfect number.

<u>Remark I</u>: A question can be raised if k is prime by itself $\Rightarrow$ $2^{k-1}(2^k-1)$ is a perfect number. The answer is negative as it will be easily shown that it does not work for k=11.

Corollary 1: : If 2^k-1 is prime, then $n = 2^{k-1} + 2^k + 2^{k+1} \ldots + 2^{2k-2}$ is a perfect number.

Proof: We have:

$$n = 2^{k-1} + 2^k + 2^{k+1} \ldots + 2^{2k-2} = 2^{k-1}\left(1 + 2 + 2^2 + 2^3 \ldots + 2^{k-1}\right)$$

$$n = 2^{k-1}\left(2^k - 1\right)$$

$\Rightarrow$ n is a perfect number by Theorem 1.

Remark III: Every even perfect number n is of the form $n = 2^{k-1}(2^k-1)$. We will not prove this, but we will accept and use it. So, the converse to Theorem 1 is also true. This is called Euler's Theorem.

Next we will show how **Remark III** applies to the first four perfect numbers. Note that:

$$6 = 2 \cdot 3 = 2^1\left(2^2 - 1\right) = 2^{2-1}\left(2^2 - 1\right)$$
$$28 = 4 \cdot 7 = 2^2\left(2^3 - 1\right) = 2^{3-1}\left(2^3 - 1\right)$$
$$496 = 16 \cdot 31 = 2^4\left(2^5 - 1\right) = 2^{5-1}\left(2^5 - 1\right)$$
$$8128 = 64 \cdot 127 = 2^6\left(2^7 - 1\right) = 2^{7-1}\left(2^7 - 1\right)$$

The characteristics of perfect numbers include, the use of Mersenne Primes which is a subclass of prime numbers. $(2^n - 1)$ It must be a positive integer ranging from 0 to positive infinity. Perfect numbers are equal to its aliquot sums. This means it is the sum of its proper factors. Lastly, all perfect numbers that have been found end in the numbers 6 or 8, so being even is a characteristic.

An example in simpler terms would go as follows: if you start with the number 6, the first question that you must ask yourself is, "What are the proper divisors of 6?" Your answer would be 1, 2, and 3. The next question to ask yourself would be, "If I added all of these numbers up, would the aliquot sum be equal to the number that I started with?"

$$1 + 2 + 3 = 6$$

Starting number = 6

Aliquot Sum = 6

Therefore, the number 6 is a perfect number.

Some non-examples of Perfect Numbers go as follows:

Given number: 8
Proper Divisors: 1, 2, 4
Aliquot Sum: $1 + 2 + 4 = 7$
The aliquot sum does **not** equal the starting number, therefore 8 is **not a perfect number**

Starting numbers that are not equal to the aliquot sum are either deficient or abundant. The number eight would be considered as a deficient number because the aliquot sum is less than the number we started with.

Another non-example would be the number thirty

Given number: 30
Proper Divisors: 1, 2, 3, 5, 6, 10, 15
Aliquot Sum: $1 + 2 + 3 + 5 + 6 + 10 + 15 = 42$
The aliquot sum does **not** equal the starting number, therefore 8 is **not a perfect number**

In this case, the number thirty would be considered an abundant number since the aliquot sum is greater than the given number.

As easy way to remember this is:
Deficient – Perfect - Abundant
Less – Equal - More

If you do not want to go through every single number, you can just use the formula given above.

If $2^{k}-1$ (k>1) is prime, then $n = 2^{k-1}(2^{k}-1)$ is a perfect number.

The largest known perfect number was discovered by a computer volunteer at The Greats Internet Mersenne Prime Search (GIMPS). It would take about 6,000 pages of paper to print this perfect number and has more than 22 million digits.

$$2^{82589932}(2^{82589933} - 1)$$

If you can find the next prime number, you can find the next Mersenne prime number, and if you can find the next Mersenne prime number, you can find the next Perfect Number. If you do end up being the next person, you will be rewarded with $150,000 if the number has at least 100 million digits and $250,000 if the number has at least 1 billion digits.

So, the next big questions is, Who's Ready to get Paid?

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