Oscillation criteria for a class of fractional dynamic equations on time scales

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Abstract: In this paper, we investigate oscillation for a class of fractional dynamic equations with damping term on time scales, and establish some oscillation criteria for it. The established oscillation criteria unify continuous and discrete analysis, and are new results so far in the literature.

Key–Words: oscillation; fractional dynamic equations; time scales; conformable fractional derivative

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1 Introduction

In [1], Hilger initiated the theory of time scale trying to treat continuous and discrete analysis in a consistent way. Based on the theory of time scale, Many authors have taken research in oscillation of various dynamic equations on time scales (see [2-8] for example). In these investigations for oscillation of dynamic equations on time scales, we notice that most of the results are concerned of dynamic equations involving derivatives of integer order, while none attention has been paid to the research of oscillation of fractional dynamic equations on time scales so far in the literature.

In this paper, we will establish some new oscillation criteria for the following conformable fractional dynamic equation with damping term on time scales of the following form:

$$(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)}+p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}+q(t)x(t)=0,\ t\in\mathbb{T}_0, \tag{1.1}$$

where $\alpha\in(0,1]$, $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0=[t_0,\infty)\bigcap\mathbb{T}$, $t_0>0$, $a,\ r,\ p,\ q\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$. For the sake of convenience, denote $\delta_1(t,t_i)=\int_{t_i}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s$, where $\widetilde{p}(t)=t^{\alpha-1}p(t)$.

2 Main Results

Theorem 2.1. Assume $-\frac{p}{a}\in\mathfrak{R}_+$, and $\int_{t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s=\infty$, $\int_{t_0}^{\infty}\frac{1}{r(s)}\Delta^\alpha s=\infty$,
$\lim\sup_{t\to\infty}\int_{t_0}^{t}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]\Delta^\alpha\xi=\infty$.
Define $\mathbb{D}=\{(t,s)|t\geq s\geq t_0\}$. If there exists a function $H\in C_{rd}(\mathbb{D},\mathbb{R})$ such that

$$H(t,t)=0,\ for\ t\geq t_0,\ \ H(t,s)>0,\ for\ t>s\geq t_0, \tag{2.1}$$

and H has a nonpositive continuous $\alpha-$ partial fractional derivative $H_s^{(\alpha)}(t,s)$ with respect to the second variable, and

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\{\int_{t_0}^{t} H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \phi(s)(a(s)\varphi(s))^{(\alpha)} + \frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)} - \frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s\} = \infty, \tag{2.2}$$

where t_2 is sufficiently large. Then every solution of Eq. (1.1) is oscillatory or tends to zero.

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By [9, Theorem 2.1 (ii)] we have either $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ for some sufficiently large t_2 or $\lim_{t\to\infty} x(t)=0$.

Now we assume $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$. define the generalized Riccati function:

$$\omega(t) = \phi(t)a(t)[\frac{(r(t)x^{(\alpha)}(t))^{(\alpha)}}{x(t)e_{-\frac{\tilde{p}}{a}}(t,t_0)} + \varphi(t)].$$

Then by [9, Theorem 2.1 (i)] one has $\omega(t)\geq 0$. Furthermore, by Theorem 1.12 (ii), Theorem 1.11 and Theorem 2.2 in [9] one can deduce that

$$\omega^{(\alpha)}(t) \leq -q(t)\frac{\phi(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \phi(t)[a(t)\varphi(t)]^{(\alpha)} - \frac{\phi(t)\delta_1(t,t_2)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)} + \frac{[\phi^{(\alpha)}(t)r(t)+2\phi(t)\delta_1(t,t_2)a(\sigma(t))\varphi(\sigma(t))]^2}{4r(t)\phi(t)\delta_1(t,t_2)}.$$

Moreover, we have

$$q(t)\frac{\phi(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} - \phi(t)(a(t)\varphi(t))^{(\alpha)} + \frac{\phi(t)\delta_1(t,t_2)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)} - \frac{[\phi^{(\alpha)}(t)r(t)+2\phi(t)\delta_1(t,t_2)a(\sigma(t))\varphi(\sigma(t))]^2}{4r(t)\phi(t)\delta_1(t,t_2)} \leq -\omega^{(\alpha)}(t). \tag{2.3}$$

Substituting t with s in (2.3), multiplying both sides by $H(t,s)$ and fulfilling α-fractional integral with respect to s from t_2 to t, together with the properties of conformable fractional calculus one can obtain that

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)\{q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \phi(s)(a(s)\varphi(s))^{(\alpha)} + \frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)} \\
&- \frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\Delta^\alpha s \\
&\leq -\int_{t_2}^{t} H(t,s)\omega^{(\alpha)}(s)\Delta^\alpha s = H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H_s^{(\alpha)}(t,s)\omega(\sigma(s))\Delta^\alpha s \leq H(t,t_2)\omega(t_2) \leq H(t,t_0)\omega(t_2),
\end{aligned}$$

where in the last two steps we have used the fact that the function $H(t,s)$ is decreasing with respect to the second variable due to $H_s^{(\alpha)}(t,s)$ is nonpositive. Then

$$\begin{aligned}
&\int_{t_0}^{t} H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \phi(s)(a(s)\varphi(s))^{(\alpha)} + \frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)} \\
&- \frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s \\
&= \int_{t_0}^{t_2} H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \phi(s)(a(s)\varphi(s))^{(\alpha)} + \frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)} \\
&- \frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s \\
&+ \int_{t_2}^{t} H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \phi(s)(a(s)\varphi(s))^{(\alpha)} + \frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}
\end{aligned}$$

$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s$$
$$\leq H(t,t_0)\omega(t_2)+H(t,t_0)\int_{t_0}^{t_2}|q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}|\Delta^\alpha s.$$

Furthermore,

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s$$
$$\leq \omega(t_2)+\int_{t_0}^{t_2}|q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}|\Delta^\alpha s<\infty,$$

which contradicts (2.2), and then the proof is completed.

Theorem 2.2. Under the conditions of Theorem 2.1. If either of the following two conditions satisfy:

$$(i).\ \lim_{t\to\infty}\sup\frac{1}{(t-t_0)^m}\{\int_{t_0}^{t}(t-s)^m[q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s\}=\infty,\ m\geq 1, \tag{2.4}$$
$$(ii).\ \lim_{t\to\infty}\sup\frac{1}{(\ln t-\ln t_0)}\{\int_{t_0}^{t}(\ln t-\ln s)[q(s)\frac{\phi(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s\}=\infty, \tag{2.5}$$

then every solution of Eq. (1.1) is oscillatory or tends to zero.

The proof of Theorem 2.2 can be reached by setting $H(t,s)=(t-s)^m,\ m\geq 1$ or $H(t,s)=\ln\frac{t}{s}$ in Theorem 2.1.

Remark. In the established oscillation criteria above, if we set $\alpha=1$, then the results reduce to corresponding oscillation criteria for dynamic equations on time scales involving integer order derivative.

References:

[1] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, Results Math., 18 (1990) 18-56.

[2] Z. Han, T. Li, S. Sun and F. Cao, Oscillation criteria for third order nonlinear delay dynamic equations on time scales, Ann. Polon. Math., 99 (2010) 143-156.

[3] T. S. Hassan, Oscillation criteria for higher order quasilinear dynamic equations with Laplacians and a deviating argument, J. Egypt. Math. Soc., 25 (2) (2016) 178-185.

[4] L. Erbe and T. S. Hassan, Oscillation of Third Order Nonlinear Functional Dynamic Equations on Time Scales, Diff. Equ. Dynam. Sys., 18 (1) (2010) 199-227.

[5] Y. B. Sun, Z. Han, Y. Sun and Y. Pan, Oscillation theorems for certain third order nonlinear delay dynamic equations on time scales, Electron, J. Qual. Theory Differ. Equ., 75 (2011) 1-14.

[6] T. Li and S. H. Saker, A note on oscillation criteria for second-order neutral dynamic equations on isolated time scales, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 4185-4188.

[7] R. P. Agarwal, M. Bohner, Li, C. Zhang, Oscillation criteria for second-order dynamic equations on time scales, Appl. Math. Lett. 31 (2014) 34-40.

[8] M. Bohner, T. S. Hassan, T. Li, Fite-Hille-Wintner-type oscillation criteria for second-order halflinear dynamic equations with deviating arguments, Indagationes Mathematicae 29(2) (2018) 548-560.

[9] B. Zheng, Some results for a class of conformable fractional dynamic equations on time scales, IJRDO-Journal of Educational Research, 4(5) (2019) 55-60.