**Triangular Numbers in Quadratic Functions Form, Generating Functions and Continued Fractions**

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**ABSTRACT**

The $n$th triangular number denoted by $T_n$ is defined as the sum of the first $n$ consecutive positive integers, and a positive integer $n$ is a triangular number if and only if $T_n = \frac{n(n+1)}{2}$. In this paper we represent a triangular number by a quadratic function i.e., for each $m \in \mathbb{Z}$ the necessary and sufficient condition for a quadratic function $f(x) = x^2 + x - 2m$ to be triangular is proved. We also prove a theorem associated to a rational root $d$ of a quadratic function $f(x)$ to be a triangular number $T_n$. We also use Generating function to represent the sets of Quotients of triangular numbers.

**KEYWORDS:** Triangular Numbers, Quadratic functions, Sequences and Factorials

**INTRODUCTION**

A triangular number $T_n$ is a number of the form $T_n = 1 + 2 + 3 + \cdots + n$, where $n$ is a natural number. For instance, the first few triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45 [1, 2, 3]. A well known fact about triangular numbers is that $y$ is a triangular number if and only if $(8y + 1)$ is a perfect square. Triangular numbers can be thought of as the numbers of dots that can be arranged in the shape of a triangle. Another interesting aspect of the triangular numbers is that they are in consecutive pairs of alternating odd and even integers. The table of triangular numbers (pages 6 and 7) illustrates this fact.

**Lemma 0.0.1:** A positive integer $k$ is called Triangular if and only if there exists a positive integer $n$ such that $k = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = T_n \ [1, 4, 5, 6].$

**Example 0.0.2** Prove that $25k + 3$ is triangular if $k$ is triangular.

**Proof:** We show that $25k + 3 = \frac{x(x+1)}{2}$ for some $x \geq 1$. Suppose $k$ is triangular. By (Lemma 0.0.1), for some $n \geq 1, k = \frac{n(n+1)}{2}$. Hence, $25k + 3 = 25\left(\frac{n(n+1)}{2}\right) + 3 = \left(\frac{25n^2 + 25n + 6}{2}\right) = \frac{5n^2 + 5n + 3}{2} = \frac{x(x+1)}{2}$. Therefore $25k + 3$ is triangular.

**Theorem 0.0.3** A positive integer $m$ is a triangular number if and only if an odd root $d$ of a quadratic function $f(x) = x^2 + x - 2m$ divides $m$.

**Proof:** ($\Rightarrow$) Suppose a positive integer $m$ is triangular and $d$ is an odd root of $f(x)$. We show that $d|m$. There exists $n \in \mathbb{Z}^+$ such that $m = \frac{n(n+1)}{2}$ (Lemma 0.0.1). This implies $f(x) = x^2 + x - 2 = x^2 + x - 2 \frac{n(n+1)}{2} = x^2 + x - n(n + 1)$. Because $d$ is a root of $f(x)$ we have $f(d) = d^2 + d - n(n + 1) = 0$. 

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Proof: Using quadratic formula, we have

\[ d = \frac{-1 \pm \sqrt{1^2 - 4(1)(n(n+1))}}{2} = \frac{-1 \pm \sqrt{1 - 8n}}{2} \]

This implies \( d = \frac{-1 + 2(n+1)}{2} \), or \( d = \frac{-1 - 2(n+1)}{2} \). That is, \( d = n \) or \( d = -(n + 1) \).

We consider two cases. First \( m = \frac{n(n+1)}{2} \) when \( n \) is even i.e., \( n = 2k \) for some \( k \in \mathbb{Z}^+ \). This implies \( m = \frac{2k(2k+1)}{2} = k(2k + 1) \), and then \( d = -(n + 1) = -(2k + 1)m \). Second when \( n \) is odd i.e., \( n = 2k + 1 \) for some \( k \in \mathbb{Z}^+ \). We have \( m = \frac{(2k+1)(2k+2)}{2} = (2k + 1)(k + 1) \) and \( d = n = (2k + 1)m \).

\[ (\Rightarrow) \text{ Suppose an odd root } d \text{ of } f(x) = x^2 + x - 2m \text{ divides } m. \text{ We show that } m \text{ is triangular. As } d \text{ is a root of } f(x) = x^2 + x - 2m \text{ it follows } f(d) = d^2 + d - 2m = 0, \text{ and } d \text{ divides } m \text{ implies } m = dc \text{ for some } c \in \mathbb{Z}^+. \text{ Combining the former and later we have } f(d) = d^2 + d - 2dc = 0. \text{ Therefore, } d^2 + d - 2dc = d(d + 1 - 2c) = 0, \text{ and either } d = 0 \text{ or } (d + 1 - 2c) = 0. \text{ As } d \text{ divides } m, d \neq 0. \text{ This implies that } (d + 1 - 2c) = 0, \text{ and } 2c = d + 1, \text{ and } c = \frac{d + 1}{2}. \text{ Thus, } m = dc = \frac{d(d + 1)}{2} \text{ and hence } m \text{ is triangular.} \]

**Theorem 0.0.4** All roots of a quadratic function \( f(x) = x^2 + x - 2m \) are rational if and only if \( m \) is triangular.

**Proof.** \( (\Rightarrow) \) Suppose a quadratic function \( f(x) = x^2 + x - 2m \) has rational root \( d \). Then the root \( d = \frac{-1 \pm \sqrt{1 + 8m}}{2} \) is rational. This implies the discriminant \( D = (1 + 8m) \) must be a perfect square. If \( (1 + 8m) \) is a perfect square, then there exists an integer \( p \) such that \( p^2 = 1 + 8m \). But \( 1 + 8m = 1 + 2(4m) = 1 + 2t \) for some \( t = 4m \in \mathbb{Z}^+ \) and is an odd integer. Consequently \( p^2 \) is odd and \( p \) is odd too. This implies there is \( a \in \mathbb{Z}^+ \) such that \( p = 2a + 1 \) and \( (2a + 1)^2 = 1 + 8m \). Hence \( 4a^2 + 4a + 1 = 1 + 8m \) and \( 4a^2 + 4a = 8m \). Consequently, \( 4a(a + 1) = 8m \) and then \( m = \frac{a(a + 1)}{2} \). Therefore \( m \) is a triangular number.

Suppose \( m \) is a triangular number and a quadratic function \( f(x) = x^2 + x - 2m \) has a real root. Then we show that it is not an irrational number. The quadratic function \( f(x) = x^2 + x - t(t + 1) \) where \( m = \frac{t(t + 1)}{2} \) is triangular implies \( f(x) = x^2 + x - t(t + 1) = (x - t)(x + t + 1) = 0 \) has a root \( x \) where either \( x = t \) or \( x = -(t + 1) \) which is a rational number.

**Corollary 0.0.5** If a quadratic function \( f(x) = x^2 + x - 2m \) has a root \( t \), then \( m = \frac{t(t + 1)}{2} \).

**Proof:** Suppose a quadratic function \( f(x) = x^2 + x - 2m \) has a root \( t \). The \( f(t) = t^2 + t - 2m = 0. \) This implies \( t^2 + t = 2m \) and \( t(t + 1) = 2m \), consequently \( m = \frac{t(t + 1)}{2} \).
Example 0.0.6 Consider the quadratic function \( f(x) = x^2 + x - 30 \). Then \( f(x) = (x + 6)(x - 5) = 0 \) implies the roots of \( f(x) \) are \( d = -6 \) or \( d = 5 \).

Consequently, \( m = \frac{d(d+1)}{2} = \frac{5(5+1)}{2} = \frac{6(-6+1)}{2} = 15 \), \( T_5 \) is a triangular number.

Theorem 0.0.7 Let \( f_i(x) = x^2 + x - 2T_i \) and \( P(x) = \prod_{i=1}^{n} f_i(x) \) where \( T_i \) and \( R_i \) are triangular numbers and roots to \( f_i(x) \) respectively for each \( i \geq 1 \). Then

1) \( \deg P(x) = 2n \), and
2) \( \prod_{i=1}^{2n} R_i = (-1)^n 2^n \prod_{i=1}^{n} T_i \)

Proof: 1) Given \( f_i(x) = x^2 + x - 2T_i \) where \( T_i = \frac{i(i+1)}{2} \). Then \( \deg f_i(x) = 2 \) for each \( i \geq 1 \).

For nonzero polynomial functions \( f(x) \), \( h(x) \), and \( g(x) \) such that \( f(x) = h(x)g(x) \),
\[ \deg f(x) = \deg h(x) + \deg g(x). \]

Hence \( \deg (P(x)) = \deg (\prod_{i=1}^{n} f_i(x)) = \sum_{i=1}^{n} \deg f_i(x) = \sum_{i=1}^{n} 1 = 2n \).

Consider \( f_i(x) = x^2 + x - 2T_i = x^2 + x - i(i+1) = (x + (i + 1))(x - i) \). This implies \( f_i(x) = 0 \) if and only if \((x + (i + 1))(x - i) = 0 \) if and only if \( x = i \) or \( x = -(i + 1) \). Set \( R_i = i \) or \( R_i = -(i + 1) \). Each quadratic polynomial function \( f_i(x) \) has two distinct roots. This implies the product of all roots of the polynomial \( P(x) \),
\[ \prod_{i=1}^{2n} R_i = \prod_{i=1}^{n} -(i + 1)(i) = \prod_{i=1}^{n} -i(i + 1) \prod_{i=1}^{n} i = (-1)^n (n + 1)! \cdot (n)!. \]  

But \( \prod_{i=1}^{n} T_i = \prod_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2^n} \prod_{i=1}^{n} i(i+1) = \frac{1}{2^n} (n!) (n+1)! \). This implies
\[ 2^n \prod_{i=1}^{n} T_i = n!(n+1)! \]

Combing (⊙) and (⊙⊙) we have \( \prod_{i=1}^{2n} R_i = (-1)^n 2^n \prod_{i=1}^{n} T_i \).

Define a sequence,

- \( \{a_i\}_{i=1}^{\infty} = \{1, \frac{1}{3}, \frac{1}{5}, \frac{2}{7}, \frac{5}{9}, \ldots\} = \{T_i\}_{i=1}^{\infty} = \{b_i\}_{i=1}^{\infty} \cup \{c_i\}_{i=1}^{\infty} \) where

- \( \{b_i\}_{i=1}^{\infty} = \{\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \ldots\} = \{\frac{2i-1}{2i+1}\}_{i=1}^{\infty} = \{\frac{T_{2i-1}}{2i}\}_{i=1}^{\infty} = \{f_i\}_{i=1}^{\infty} \) and

- \( \{c_i\}_{i=1}^{\infty} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \)

\[ \left( \frac{T_{2i}}{2i}\right)_{i=1}^{\infty} = \left( \frac{h_i}{l_i}\right)_{i=1}^{\infty} \quad \text{and} \quad \gcd(f_i, g_i) = \gcd(h_i, l_i) = 1, [7] \]

Set:

- \( a) \{d_i\}_{i=1}^{\infty} = \{1, 1, 3, 2, 5, 3, 7, 4, \ldots\} = \bigcup_{i=1}^{\infty} \{2i - 1, i\} \quad \text{and} \)
- \( b) R_i = \{i(2i - 1), i(2i + 1) : i \geq 1\} \)
- \( c) \text{ For each } i \geq 0, \left\{ \begin{array}{l}
S_{2i} = 2i + 1 \\
S_{2i+1} = i + 1
\end{array} \right. \)
Theorem 0.0.8  Define for each $i \geq 0$,
\[
\begin{cases}
  s_{2i} = 2i + 1, \\
  s_{2i+1} = i + 1,
\end{cases}
\]
Then $T_i = S_{i-1} \ast S_i$ is a triangular number for each $i \geq 1$.

Proof:

Case 1: We first considered the case when $i$ is even, i.e., $i = 2k$.

Then $T_{2k} = T_i = S_{i-1} \ast S_i$
\[
= S_{2k-1} \ast S_{2k} = S_{2(k-1)+1} \ast S_{2k},
\]
because $2k - 1 = 2(k - 1) + 1$

\[
= ((k - 1) + 1) \ast (2k + 1) \quad \text{by} \ (*) \ \text{and} \ (**)
\]
\[
= k \ast (2k + 1) = \frac{2k \ast (2k + 1)}{2} = \frac{i \ast (i+1)}{2}
\]

Therefore, $T_i = \frac{i \ast (i+1)}{2}$, and by (Theorem 0.0.1) $T_i = S_{i-1} \ast S_i$ is a triangular number.

Case 2: Now we considered the case when $i$ is odd, i.e., $i = 2k + 1$.

This implies that $T_{2k+1} = T_i = S_{i-1} \ast S_i$
\[
= S_{2k+1-1} \ast S_{2k+1}
\]
\[
= S_{2k} \ast S_{2k+1},
\]
by ($*$) and ($**$)

\[
= \frac{(2k+1) \ast (2k+1+1)}{2} = \frac{i \ast (i+1)}{2}
\]

Therefore, $T_i = \frac{i \ast (i+1)}{2}$, and by (Theorem 0.0.1), $T_i = S_{i-1} \ast S_i$ is a triangular number. $\blacksquare$

Corollary 0.0.9 [8] [A105658] off set {0}

The set $F = \bigcup_{i=1}^{\infty} \{ 2i - 1, \ i \}$ is the set of integers that satisfies the statement of (Theorem 0.0.10).

$F = \{1,1,3,2,5,3,7,4,9,\ldots\}$

Theorem 0.0.10 Set $R_i = \{ h_i f_i = i(2i - 1), \ h_i g_i = i(2i + 1): i \geq 1 \}$.

Then $\bigcup_{i=1}^{n} R_i = \bigcup_{i=1}^{2n} T_i$ (\textcircled{1})
Proof: Denote \( \eta_i = h_if_i = i(2i - 1) \) and \( \mu_i = h_ig_i = i(2i + 1) \) for each \( i \geq 1 \). Then \( R_i = \{ \eta_i, \mu_i | i \geq 1 \} \).

We set \( F_n = \bigcup_{i=1}^{n} \eta_i \) and \( G_n = \bigcup_{i=1}^{n} \mu_i \). This implies \( \bigcup_{i=1}^{n} R_i = F_n \cup G_n \).

But \( \eta_i = i(2i - 1) = \frac{(2i-1)(2i)}{2} = T_{2i-1} \), \( i \geq 1 \) and \( \mu_i = i(2i + 1) = \frac{(2i)(2i+1)}{2} = T_{2i} \), \( i \geq 1 \),

are triangular numbers [7]. We use induction to prove the statement. We verify it is true for \( n = 1 \).

The left side of \( \bigcirc \bigcirc \), \( R_1 = F_1 \cup G_1 = \eta_1 \cup \mu_1 = \{ T_1, T_2 \} = \bigcup_{i=1}^{1} R_i \) and the right side \( \bigcup_{i=1}^{2} T_i \) are equal. Hence true for \( n = 1 \). Let \( k \in \mathbb{Z}^+ \) and suppose the statement in \( \bigcirc \bigcirc \) is true for \( n = k \) that is \( \bigcup_{i=1}^{k} R_i = \bigcup_{i=1}^{2k} T_i \).

Now we show that it is true for \( k = n + 1 \). Thus \( \bigcup_{i=1}^{k+1} R_i = \bigcup_{i=1}^{k} R_i \cup \{ R_{k+1} \} = \bigcup_{i=1}^{2k} T_i \cup \{ R_{k+1} \} = \bigcup_{i=1}^{2k} T_i \cup \{ \eta_{k+1}, \mu_{k+1} \} \).

From \( \bigcirc \bigcirc \) \( \eta_{k+1} = T_{2(k+1)-1} = T_{2k+1} \) and \( \mu_{k+1} = T_{2(k+1)} = T_{2k+2} \). This implies \( \bigcup_{i=1}^{2k} T_i \cup \{ \eta_{k+1}, \mu_{k+1} \} = \bigcup_{i=1}^{2k} T_i \cup \{ T_{2k+1}, T_{2k+2} \} = \bigcup_{i=1}^{2(k+1)} T_i \) and \( \bigcup_{i=1}^{k+1} R_i = \bigcup_{i=1}^{2(k+1)} T_i \).

This implies the statement is true for \( k = n + 1 \).

Hence the statement in \( \bigcirc \bigcirc \) is true \( \forall k \geq 1 \).

Definition 0.0.11
A finite or infinite expression of the form

\[
a = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}} \quad (*)
\]

where the \( a_i \) are real numbers with \( a_1, a_2, a_3, a_4, \ldots > 0 \) is called a continued fraction. The numbers \( a_i \) are called the partial quotients of the continued fraction.

The continued fraction \( (*) \) is called simple if partial if the partial quotients \( a_i \) are all integers. It is called finite if it terminates, i.e., if it is of the form

\[
a = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}} \quad (**) \]

and infinite otherwise. [9, 10, 11]
**Notation:** (Bracket notation for continued fractions). The continued fractions \((\ast)\) and \((\ast \ast)\) are denoted by \([a_1; a_2, a_3, a_4, \ldots]\) and \([a_1; a_2, a_3, a_4, \ldots, a_n]\) respectively.

**Example 0.0.12**

\[ a = 1 + \cfrac{1}{1 + [1; 2, 2, 2, \ldots]} = 1 + \cfrac{1}{1+a} \]

Rearranging, we see \(a\) must be a solution of \(x^2 = 2\), but since \(a\) is a positive (Indeed \(a > 1\)), we have \(a = \sqrt{2}\).

**Theorem 0.0.13** [12]

I. Continued fraction, \(x = \cfrac{f_1}{g_1 + \cfrac{f_2}{g_2 + \cfrac{f_3}{g_3 + \cdots}}} = \cfrac{1}{e-1}\) \((\ast)\)

II. Let \(a = \{f_1, g_1, f_2, g_2, f_3, g_3, \ldots\}\) and \(b = \{h_1, l_1, h_2, l_2, h_3, l_3, \ldots\}\). The two set of numbers are generated by, \(GF, f(x) = \cfrac{1}{(1-x)(1-x^2)^2}\) and \(g(x) = \cfrac{1+x+x^3+x^3}{(1-x^2)^2}\) respectively.

Some ODD and EVEN Triangular Numbers with Corresponding Subscripts [13]

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<td>4*9</td>
<td>6*11</td>
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<td>8*15</td>
<td>8*17</td>
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<td>10*21</td>
<td>12*23</td>
<td>12*25</td>
<td>13*27</td>
<td>13*29</td>
</tr>
</tbody>
</table>

\(\{t_3, t_4, t_7, t_8, t_{11}, t_{12}, t_{15}, t_{16}, t_{19}, t_{20}, t_{23}, t_{24}, t_{27}, t_{28}\}\)

The table above shows even **triangular numbers** with their respective T subscripts (see shaded)

\[ \begin{cases} t_{2i-2}, & \text{i is even} \\ t_{2i-1}, & \text{i is odd} \end{cases} \quad \Rightarrow \quad \begin{cases} t_{4k-2}, & \text{for } i = 2k, k \in \mathbb{Z}^+ \\ t_{4k-3}, & \text{for } i = 2k - 1, k \in \mathbb{Z}^+ \end{cases} \]
The table above shows odd triangular numbers with their respective T subscripts (see shaded).

\[
\begin{align*}
\begin{array}{cccccccccccccccc}
1 & 3 & 15 & 21 & 45 & 55 & 91 & 105 & 171 & 231 & 253 & 325 & 351 \\
1 \times 1 & 1 \times 3 & 3 \times 5 & 5 \times 7 & 5 \times 11 & 7 \times 13 & 7 \times 15 & 9 \times 17 & 9 \times 19 & 11 \times 21 & 11 \times 23 & 13 \times 25 & 13 \times 27 \\
t_1 & t_2 & t_5 & t_6 & t_9 & t_{10} & t_{13} & t_{14} & t_{17} & t_{21} & t_{22} & t_{25} & t_{26}
\end{array}
\end{align*}
\]

\textbf{Theorem 0.0.11:} Any two consecutive even triangular numbers have the form \( T_{4k} \) and \( T_{4k-1} \) for each \( k \geq 1 \), and \( \sum_{i=1}^{n}(T_{4i}^2 - T_{4i-1}^2) = 64T_n^2 \). Likewise any two consecutive odd triangular numbers are \( T_{4k-2} \) and \( T_{4k-3} \) for each \( k \geq 1 \) and

\[
\sum_{i=1}^{n}(T_{4i-2}^2 - T_{4i-3}^2) = 8T_{2n^2-1}^2 .
\]

\textbf{Proof:} We prove the statement when

1) the triangular numbers have even parity,

\[
T_{4i} = \frac{4i(4i+1)}{2} \quad \text{and} \quad T_{4i-1} = \frac{(4i-1)(4i)}{2} \quad \text{by Lemma (0.0.1). This implies,}
\]

\[
T_{4i}^2 - T_{4i-1}^2 = \left( \frac{4i(4i+1)}{2} \right)^2 - \left( \frac{(4i-1)(4i)}{2} \right)^2 = \left( \frac{4i}{2} \right)^2 ((4i + 1)^2 - (4i - 1)^2) = 4i^2(4i) = (4i)^3
\]

As \( T_n = \frac{n(n+1)}{2} \) for each \( n \geq 1 \) and \( \sum_{i=1}^{n}i^3 = \frac{n^2(n+1)^2}{4} \) we have

\[
\sum_{i=1}^{n}(T_{4i}^2 - T_{4i-1}^2) = \sum_{i=1}^{n}(4i)^3 = 64 \sum_{i=1}^{n}i^3 = 64 \frac{n^2(n+1)^2}{4} = 64 \left( \frac{n(n+1)}{2} \right)^2 = 64T_n^2 .
\]

2) the triangular numbers have odd parity,

\[
T_{4i-2} = \frac{(4i-2)(4i-1)}{2} \quad \text{and} \quad T_{4i-3} = \frac{(4i-3)(4i-2)}{2} \quad \text{(Lemma 0.0.1). This implies}
\]

\[
T_{4i-2}^2 - T_{4i-3}^2 = \left( \frac{(4i-2)(4i-1)}{2} \right)^2 - \left( \frac{(4i-3)(4i-2)}{2} \right)^2 = \left( \frac{(4i-2)}{2} \right)^2 ((4i - 1)^2 - (4i - 3)^2)
\]

\[
= (2i - 1)^2(16i - 2) = 8(2i - 1)^3 .
\]

Therefore, \( \sum_{i=1}^{n}(T_{4i-2}^2 - T_{4i-3}^2) = \sum_{i=1}^{n}8(2i - 1)^3 = 8 \sum_{i=1}^{n}(2i - 1)^3 . \) Let \( 2i - 1 = k . \)

Hence \( \sum_{i=1}^{n}(T_{4i-2}^2 - T_{4i-3}^2) = 8 \sum_{k=1}^{2n^2-1}k^3 = 8T_{2n^2-1}^2 . \) \( \square \)
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