Abstract: We will investigate which integers can be written as the sum of squares. Different examples are given to supplement each given theorem.

Introduction: We say that a positive integer \( n \) is representable as a sum of two squares if \( n = a^2 + b^2 \) for some integers \( a \) and \( b \). We include 0 as a possible value of \( a \) and \( b \). We also say that a positive integer \( n \) is representable as a sum of \( m \) squares if \( n = a_1^2 + a_2^2 + a_3^2 + \cdots + a_m^2 \) for some integers \( m \) and \( a_i \).

1. The Sum of Two Squares

Theorem 1. An integer \( n \) is the sum of two squares \( \iff \) \( 2n \) is the sum of the squares.

Proof (1) \( \Rightarrow \) Assume \( n \) is the sum of two squares. Let \( n = a^2 + b^2 \) for integers \( a \) and \( b \).

Then \( 2n = 2(a^2 + b^2) \)
\( \Rightarrow 2n = (a + b)^2 + (a - b)^2 \)
\( \Rightarrow 2n \) is the sum of two squares

(2) \( \Leftarrow \) Assume \( 2n = c^2 + d^2 \). Since \( c \) and \( d \) are both even or both odd \( c + d \) and \( c - d \) are even integers.

\[ n = \left( \frac{c + d}{2} \right)^2 + \left( \frac{c - d}{2} \right)^2 \]
\( \Rightarrow n \) is the sum of two squares.

The theorem follows by (1) and (2).
**Example 1.** Let \( n = 29 \) then

\[
\begin{align*}
 n^2 &= 5^2 + 2^2 \\
 2n &= 58 = 7^2 + 3^2
\end{align*}
\]

**Theorem 2.** If \( n \) a triangular number, prove that even if each of the three consecutive integers \( 8n^2, 8n^2 + 1 \), and \( 8n^2 + 2 \) can be expressed as a sum of two squares.

**Proof**

1) \( 8n^2 = (2n)^2 + (2n)^2 \), hence sum of two squares

2) \( n \) is a triangular number

\[
\Rightarrow n = \frac{m(m+1)}{2}
\]

\[
\Rightarrow 8n = 4m(m + 1)
\]

\[
\Rightarrow 8n = 4(m)(m + 1) + 1
= 4m^2 + 4m + 1
= (2m + 1)^2
\]

Hence \( 8n + 1 \) is a perfect square.

Let \( 8n + 1 = k^2 \)

Now observe that

\[
2(8n^2 + 1) = (4n + 1)^2 + (8n + 1)
= (4n + 1)^2 + k^2
\]

\( \Rightarrow \) by Theorem 1, \( 8n^2 + 1 \) is a sum of two squares

3) Note that

\[
8n^2 + 2 = (m(m + 1) + 1)^2 + (m(m + 1) - 1)^2
\]

a sum of two squares also.

**Theorem 3.** If each of the natural numbers \( x \) and \( y \) is a sum of two squares then so is \( xy \).
Proof Let \( x = a^2 + b^2 \) and \( y = c^2 + d^2 \). Then

\[
xy = (a^2 + b^2)(c^2 + d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 = a^2c^2 + 2abcd + b^2d^2 + a^2d^2 - 2abcd + b^2 + c^2 = (ac + bd)^2 + (ad - bc)^2. \]

Thus the theorem is proved.

Remark 1 \( xy \) can also be written as

\[
xy = (ac - bd)^2 + (ad + bc)^2
\]

Example 2.

\[
65 = 5 \cdot 13 \quad \text{Note that}
\]

\[
5 = (2^2 + 1) \quad \text{and} \quad 13 = 3^2 + 2^2
\]

\[
a = 2 \quad b = 1 \quad c = 3 \quad d = 2
\]

So, we have

\[
65 = (6 + 2)^2 + (4 - 3)^2 = 8^2 + 1^2 = (6 - 2)^2 + (4 + 3)^2 = 4^2 + 7^2
\]

We state the following two Theorem without proof and use them.

Theorem 4. If the prime \( P \equiv 1 (mod \ 4) \) then there exist unique integers \( x \) and \( y \) such that \( x > y > 0 \) and \( p = x^2 + y^2 \).

Example 3. Let \( p = 97 \). Then \( P \equiv 1 (mod \ 4) \) and 97 can be expressed as sum of two squares. Note \( 97 = 9^2 + 4^2 \).

Theorem 5. Let \( n \) be a positive integer. Then \( n \) can be expressed as the sum of two squares if and only if all prime factors of \( n \) of the form \( 4t+3 \) have even exponents in the factorization of \( n \).

Example 4. Take \( n = 162 \). Then \( n = 2^4 \cdot 3^4 \) and 3 is a prime factor of the form \( 4t+3 \) with even exponent 4 and hence can be expressed as the sum of two squares. Note that \( 162 = 9^2 + 9^2 \).
2. The sum of three squares.

**Lemma 1:** Every number can be expressed as the sum of 3 triangular numbers.

**Theorem 6** Every number of the form $8k + 3$ can be expressed as the sum of three squares.

**Proof By** Lemma 1, $K$ can be written as the sum of three triangular numbers. That is,

$$K = \frac{a(a + 1)}{2} + \frac{b(b + 1)}{2} + \frac{c(c + 1)}{2}$$

$$\Rightarrow 8K + 3 = 4a(a + 1) + 4b(b + 1) + 4c(c + 1)$$

$$\Rightarrow 8K + 3 = 4a^2 + 4a + 4b^2 + 4b + 4c^2 + 4c + 1$$

$$= (2a + 1)^2 + (2b + 1)^2 + (2c + 1)^2$$

Hence the theorem is proved

**Remark 2:** A number can be expressed as the sum of three squares in only one way.

We state the following important theorem without proof and use it.

A natural number can be represented as the sum of three squares of integers.

$$n = a^2 + b^2 + c^2 \iff n \text{ is of the form}$$

$$n = 4^m(8k + 7) \text{ for integers } m \text{ and } k$$

**Example 5** List five integers that can be expressed as the sum of three square integers using $n = 8k + 3$

$$k = 0 \Rightarrow n = 3 = 1^2 + 1^2 + 1^2$$

$$k = 1 \Rightarrow n = 11 = 3^2 + 1^2 + 1^2$$

$$k = 2 \Rightarrow n = 19 = 2^2 + 3^2 + 1^2$$

$$k = 3 \Rightarrow n = 27 = 3^2 + 3^2 + 3^2$$
\[ k = 4 \Rightarrow n = 35 = 5^2 + 3^2 + 1^2 \]

**Theorem 7** Let \( n \) be a positive integer. Then \( n \) can be expressed as the sum of three squares if and only if \( n \) is not of the form \( 4^k \ (8t + 7) \).

**Example 6.** Let \( n = 15 \). Then 15 is of the form \( 4^k \ (8t + 7) \) and cannot be expressed as the sum of three squares.

3. The sum of four squares.

**Lagrange’s Theorem:** We state the theorem without proof and use it.

**Theorem 8** Every natural number is the sum of four squares.

**Example 4:**

\[
\begin{align*}
(1) & \quad 5 = 2^2 + 1^2 + 0^2 \\
(2) & \quad 21 = 4^2 + 2^2 + 1^2 + 0^2 \\
(3) & \quad 28 = 5^2 + 1^2 + 1^2 + 1^2 \\
\end{align*}
\]

(4) **Sum of squares of consecutive integers**

**Theorem 8** The sum of the squares of the first \( n \) natural numbers is given by

\[
\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}
\]

**Proof:** Easily follows using induction.

**Corollary 1:** The sum of the squares of the first \( n \) **even** natural numbers is given by
\[
\sum_{k=1}^{n} (2n)^2 = \frac{2n(n + 1)(2n + 1)}{3}
\]

**Corollary 2.** The sum of the squares of the first **even odd natural** numbers is given by

\[
\sum_{k=1}^{n} (2n - 1)^2 = \frac{n(2n + 1)(2n - 1)}{3}
\]

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References:

1. Petel Clark, Sum of two squares.
2. Jahnavi Bhaskar, Sum of two squares.
3. Alan Beardon, Sum of Squares and sum or cubes.