EXISTENCE THEOREM FOR ORDINARY NONLINEAR DIFFERENTIAL EQUATIONS.

S. N. Salunkhe
Dept.of Mathematics R. L. College, Parola (Dist- Jalgaon) M.S.
E-mail: saisns@rediffmail.com

Abstract: In this article, an existence theorem for ordinary nonlinear differential equations with the periodic boundary value problem is proved under mixed generalized Lipschitz and caratheodory conditions for the aspects of solutions. Our result includes some known existence results for ordinary nonlinear differential equations with the periodic boundary value problems.

Keywords: Existence theorem, ordinary nonlinear differential equations, periodic boundary value problem, Lipschitz and caratheodory conditions

1 . Second Order Periodic Boundary Value Problem.

Let $\mathbb{R}$ denote the real line. Given a closed and bounded interval $J = [O, T]$in $\mathbb{R}$. Consider the periodic boundary value problems (in short $PBVP$) of second order ordinary Differential equations with period T

$$\left.\begin{array}{l} -\frac{d^2}{dt^2}\left[\frac{x(t)}{f(t,x(t),x'(t))}\right] = g\big(t, x(t)x'(t)\big) \quad a.e.\, t \in J \\ x(O) = x(T),\ x'(O) = x'(T),\ x''(O) = x''(T) \end{array}\right\} \qquad (1.1)$$

Where $f: J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+ - \{0\}$ & $g: J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

By a solution of the $PBVP$ (1.1) we mean a function $x \in AC^1(J, \mathbb{R})$ that satisfies

i) The function $t \longrightarrow \frac{d}{dt}\left(\frac{x(t)}{f(t,x(t),x'(t))}\right)$
is absolutely continuous defined on J and

ii) x satisfies the equations in (1.1)

Where $AC^1(J, \mathbb{R})$ is the space of continuous functions whose first and second derivative exists and is absolutely continuous real valued functions defined on J. When $f(t, x, x') = 1$ for all $t \in J$ and $t \in \mathbb{R}$, the PBVP (1.1) reduces to a

$$\left.\begin{array}{l} -x''(t) = g\big(t, x(t), x'(t)\big) \quad a.e.\ \ t \in J \\ x(0) = x(T), x'(0) = x'(T)\,,\ x''(O) = x''(T) \end{array}\right\}$$

(1.2)

Where $g: J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

In the following section we describe some basic tools from nonlinear functional analysis which will be used in subsequent part of article.

2.Auxiliary Results

Let X be a Banach algebra with norm ||. ||. A mapping A: X → X is called $\mathcal{D}$-*Lipschitz* if there exists a continuous non-decreasing function $\psi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leq \psi(\|x - y\|) \tag{2.1}$$

For all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r, (\alpha > 0)$, A is called *Lipschitz* with the *Lipschitz* constant α. In particular, if $\alpha < 1$, a is called contraction with contraction constant . α Further , if $\psi(r) < \alpha$ for all $r > 0$, then A is called nonlinear $\mathcal{D}$ - contraction on X. Sometimes we call the function ψ a *D*- function of A on X for convenience.

An operator $B : X \to X$ is called compact if $\overline{B(s)}$ is a compact subset of X for any S⊂X. Similary B : $X \to X$ is called totally bounded of B maps a bounded subset of X into a relatively compact subset of X. Finally $B : X \to X$ is called completely continuous operator if it is continuous and totally bounded operator on X. A non linear alternative of Schaefer type recently proved by Dhage [3] is embodied in the following theorem.

Theorem 2.2 (Dhage [3]), Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively open and closed balls in a Banach algebra X centered at origin 0 and of radius r. $Let\ A, B : \overline{\mathcal{B}_r(0)} \to X$ be two operators satisfying.

a) A is Lipschitz with a Lipschitz constant α.
b) B is compact and continuous, and
c) $\alpha\, M < 1$, where $M = \left\|\mathcal{B}\overline{(\mathcal{B}_r(0))}\right\| := sub\left\{\|Bx\| : x\epsilon\, \overline{\mathcal{B}_r(0)}\right\}$

Then either

(i) the equation $\lambda[A_x B_x] = x$ has solution for $\lambda = 1$ or
(ii) there exists an $u \in X$ such that $\|u\| = r$ satisfying $\lambda[Ax\ Bx] = u$ For some $0 < \lambda < I$

In the following sections we prove the main existence results of this article.

3. Existence Theory

Let B(J, ℝ) denote the space of bounded real-valued functions defined on J. Let C(J, ℝ), denote the space of all continuous real-valued functions defined on J. Define a norm ||·|| and a multiplication " · " in C(J, ℝ) by

$$\|x\| = \sup_{t \in J} |x(t)|$$

And $(x \cdot y)(t) = (xy)(t) = x(t) \cdot y(t)$ for $t \in J$, clearly $C\ (J, \mathbb{R})$ becomes Banach algebra with respect to above normal and multiplication. By $L^1(J, \mathbb{R})$ we denote the vector space of Lebesgne integrable functions defined on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J, \mathbb{R})$ is defined $\|x\|_{L^1} = \int_0^T |x(t)|\, ds$.

The following lemma appears in Nieto [1] and which is useful in the study of second order periodic boundary value problems of ordinary differential equations.

Lemma 3.1 for any real number $m > 0$ and $\sigma \in L^1(J, \mathbb{R}), x$ is a solution to the differential equation

$$\left.\begin{aligned} -x''(t) + m^2 x(t) = \sigma(t) \quad & a.e. t \in J \\ x(0) = x(T), x'(0) = x'(T), x''(O) = x''(T) \end{aligned}\right\} \tag{3.2}$$

if and only if it is a solution of the integral equation $x(t) = \int_0^T G_m(t,s)\sigma(s)ds$ (3.3)

Where

$$\underset{m}{\mathrm{G}}(t,s) = \begin{cases} \frac{1}{2m(e^{mT}-1)}\left[e^{m(t-s)} + e^{m(T-t+s)}\right], 0 \le s \le t \le T, \\ \frac{1}{2m(e^{mT}-1)}\left[e^{m(s-t)} + e^{m(T-s+t)}\right] 0 \le t \le s \le T \end{cases} \tag{3.4}$$

Notice that the Green's function G_m is continuous and nonnegative on $J \times J$ and the numbers

$\alpha = min\{|G_m(t,s)| : t,s \in [0,T]\} = \frac{e^{mT}}{m(e^{mT}-1)}$ and

$\beta = max\{|G_m(t,s)| : t,s \in [0,T]\} = \frac{e^{mT}+1}{2m(e^{mT}-1)}$ exist for all positive real number .

We need the following definition in the

Definition 3.5 A mapping $\beta : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be caratheodory if

(i) $t \longmapsto \beta(t,x,x')$ is measurable for each and $x \in \mathbb{R}$ and
(ii) $x \longmapsto \beta(t,x,x')$ continuous almost everywhere for $t \in J$.

Again, a caratheodory function $\beta(t,x,x')$ is called L^1- caratheodory if

(iii) For each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that $|\beta(t,x,x')| \le h_r(t)$ $a.e\ t \in J$ for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally, a caratheodory function $\beta(t,x,x')$ is called $L^1_{\mathbb{R}}$ caratheodory if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$|\beta(t,x,x')| \le h(t)$ $a.e\ t \in J$ for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a bound function of β.

We will use the following hypotheses in the sequel.

(A_0) The functions $t \longmapsto f(t,x,x'), t \longmapsto f_t(t,x,x')$ and

$t \longmapsto f_x\,(t, x, x')$ are periodic of T all $x \in \mathbb{R}$.

(A_1) The functions $t \longmapsto \frac{x}{f(0,x,x')}$ is injective in $\mathbb{R}$.

(A_2) $f\,(o, x, x') \neq x\, f_x\,(o, x, x')$ for all $x \in \mathbb{R}$, where $f_x\,(o, x, x') = \frac{\partial f\,(t,x,x')}{\partial x}\Big|_{t=o}$

(A_3) The functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A_4) The functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell \in B\,(J, \mathbb{R})$ such that $|f\,(t, x, x') - f\,(t, y, y')| \leq \ell(t)\{|x - y| + |x' - y'|\}$ for all $t \in J$ and $x, y \in \mathbb{R}$. Moreover, we assume that $L = \max_{t \in J} \ell(t)$.

(A_5) The function g is caratheodory.

Remark 3.6 Note that hypotheses (A_3) though (A_5) are much common in the literature on the theory nonlinear differential equations. Similarly, there do exist functions satisfying the hypotheses (A_0) though (A_2). Indeed, it is easy to verify that the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f\,(t, x, x') = a + bx + cx'$ for some $a\,, b \in \mathbb{R}$ with a $\neq 0$ and $a + bx + cx' \neq 0$ satisfies the hypotheses $(\mathrm{A}_0) - (\mathrm{A}_4)$ mentioned above.

Now consider the linear perturbation of the $PBVP$ (1.1) of first type,

$$\left.\begin{aligned} -\left(\frac{x(t)}{f\big(t, x(t), x'(t)\big)}\right)'' + m^2\left(\frac{x(t)}{f\big(t, x(t), x'(t)\big)}\right) = g_m\big(t, x(t), x'(t)\big) \qquad a.e.\, t \in J \\ x(0) = x(T), x'(0) = x'(T) \end{aligned}\right\} \tag{3.7}$$

Where $m > 0$ is a real number and the function $g_m : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g_m\,(t, x, x') = g(t, x, x') + m^2\left(\frac{x}{f(t,x,x')}\right) \tag{3.8}$$

Lemma 3.9 Assume that hypotheses $(\mathrm{A}_0) - (\mathrm{A}_4)$ holds. Then for any real number $m > 0$ and $g_m\,\big(t, x(t), x'(t)\big) \in L^1\,(J, \mathbb{R})$, x is a solution to the differential equation (3.7) if and only if it is a solution of the integral equation

$$x(t) = \big[f\big(t, x(t), x'(t)\big)\big]\left(\int_0^T G_m(t, s)\, g_m\,\big(s, x(s)x'(s)\big)ds\right) \tag{3.10}$$

Where the Green's function $G_m(t, s)$ is defined by (3.4)

We make use of the following hypothesis in the sequel.

(A_6) There exists a continuous and non-decreasing function $\psi : [0, \infty) \to (0, \infty)$ and function $\gamma \in L^1(J, \mathbb{R})$ such that $\gamma(t) > 0$, $a.e.\; t \in J$ satisfying $|g_m\,(t, x, x')| \leq \gamma(t)\,\psi\,(|x|)$ $a.e.\; t \in J$

for all $x \in \mathbb{R}$.

Theorem 3.11 Assume that the hypotheses $(\mathrm{A}_0)-(\mathrm{A}_4)$ and $(\mathrm{A}_5)-(\mathrm{A}_6)$ hold. Suppose that there exists a real number $r > 0$ such that

$$r > \frac{F_0\left[\frac{e^{mT}+1}{2m(e^{\in mT}-1)}\right]\|\gamma\|_{L^1}\psi(r)}{1-L\left[\frac{e^{mt}+1}{2m(e^{\in mT}-1)}\right]\|\gamma\|_{L^1}\psi(r)} \tag{3.12}$$

Where $L\left[\frac{e^{mT}+1}{2m(e^{\in mT}-1)}\right] = \|\gamma\|_{L^1}\psi(r) < 1$ and $F_0 = \sup_{t\in[o.T]}|f(t,o)|$

Then the $PBVP$ (1.1).has solution defined on J.

Proof: Let $x = C(J,\mathbb{R})$. Defined on open ball $\mathcal{B}r(0)$ centered at origin 0 of radius r, where the real number r satisfies the inequality (3.12). Define two mappings A and B on $\mathcal{B}_r(0)$ by

$$Ax(t) = f\left(t, x(t), x'(t)\right),\ t \in J \tag{3.13}$$

and $$Bx(t) = \int_0^T G_m(t,s)\, g_m\left(s, x(s), x'(s)\right) ds,\ t \in J \tag{3.14}$$

Obviously A and B define the operators $A, B : \overline{\mathcal{B}r(0)} \to X$. Then the integral equation (3.10) is equivalent to the operator equation

$$Ax(t)\,\beta x(t) = x(t),\ t \in J \tag{3.15}$$

We shall show that the operators A and B satisfy all the hypotheses of Theorem (2.2).

We first show that A is Lipchitz on $\overline{\mathcal{B}r(0)}$.

Let $x, y \in$ X . Then by (A_3),

$$|Ax(t) - Ay(t)| \le \left|f\left(t, x(t), x'(t)\right) - f\left(t, y(t), y'(t)\right)\right|$$
$$\le \ell(t) max\{|x(t) - y(t)| + |x'(t) - y'(t)|\}$$
$$\le L\ \|x - y\|$$

for all $t \in J$. Taking the supremum over t, we obtain $\|Ax - Ay\| \le L\ \|x - y\|$

for all $x,\ y \in \overline{\mathcal{B}r(0)}$. So A is Lipschitz on $\overline{\mathcal{B}r(0)}$ with the Lipschitz constant L. Next we show that B is completely continuous on X. Using the standard arguments as in Granas et. al. [2], it is shown that B is a continuous operator on $\overline{\mathcal{B}r(0)}$. We shall show that $B\left(\overline{\mathcal{B}r(0)}\right)$ is uniformly bounded and equicontinous set in X. Let $x \in \overline{\mathcal{B}r(0)}$ be arbitrary. Since g is caratheodory, we have

$$|Bx\ (t)| \leq \left|\int_0^T G_k(t,s)\ g_m\big(s,x\ (s),x'(s)\big)\ ds\right|$$

$$\leq \left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right] \int_0^T [\gamma(s)\psi(|x(s)|)]\, ds$$

$$\leq \left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right] \int_0^T \gamma(s)\psi(|x(s)|)\, ds$$

$$\leq \left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right] \|\gamma\|_{L^1}\psi(r)$$

Taking the supremum over t, we obtain $\|Bx\| \leq M$ for all $x, \in \overline{\mathcal{B}r(0)}$, where $M = \left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right] \|\gamma\|_{L^1}\psi(r)$ This shows that $B\big(\overline{\mathcal{B}r(0)}\big)$ is a uniformly bounded set in X. Next, we show that $B\big(\overline{\mathcal{B}r(0)}\big)$ is an equi-continuous set in X. Let $x \in \overline{\mathcal{B}r(0)}$ be arbitrary. Then for any $t_1, t_2 \in J$ one has

$$|Bx(t_1) - Bx(t_2)|$$

$$\leq \int_0^T |G_m(t_1,s) - G_m(t_2,s)|\big|g_m\big(S,x(s),x'(s)\big)\big|ds$$

$$\leq \int_0^T |G_m(t_1,s) - G_m(t_2,s)|\gamma\ (s)\psi(|x(s)|)ds$$

$$\leq \int_0^T |G_m(t_1,s) - G_m(t_2,s)|\gamma\ (s)\psi(r)ds$$

$$\leq \left(\int_0^T |G_m(t_1,s) - G_m(t_2,s)|^2\right)^{1/2} \left(\int_0^T |\gamma(s)|^2\, ds\right)^{1/2} \psi(r) \qquad (3.16)$$

Hence for all $t_1, t_2 \in J$, $|Bx(t_1) - Bx(t_2)| \to 0\ as\ t_1 \longrightarrow t_2$ uniformly for all $x \in \overline{\mathcal{B}r(0)}$. There fore $B\overline{\mathcal{B}r(0)}$, is a equi-continuous set in X. Now $B\overline{\mathcal{B}r(0)}$, is a uniformly bounded and equi-continuous set in X, so it is compact by Arzela-Ascoli theorem. As a result B is compact and continuous operator on $\mathcal{B}r(0)$. Thus, all the conditions of theorem (2.2) are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be a solution to the operator equation $\lambda\,[Au\ Bu] = u$ for some $0 < \lambda < 1$ satisfying $\|u\| = r$. Then we have, for any $\lambda \in (0,1)$, $u(t) = \lambda\big[f\big(t,x(t),x'(t)\big)\big]\left(\int_0^T G_m(t,s)g_m\big(s,x(s),x'(s)\big)ds\right)$

for $t \in J$. Therefore,

$$|u(t)| \leq \lambda\big|f\big(t,u(t),u'(t)\big)\big|\left(\left|\int_0^T G_m(t,s)g_m\big(s,u(s),u'(s)\big)ds\right|\right)$$

$$\leq \lambda\big(\big|f\big(t,x(t),x'(t)\big) - f(t,0,0)\big| + |f(t,0,0)|\big) \times$$

$$\left(\int_0^T G_m(t,s)\left|g_m\big(s,u(s),u'(s)\big)\right|ds\right)$$

$$\leq [\ell(t)|u(t)| + F_0]\left(\int_0^T \left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right]\left|g_m\big(s,u(s),u'(s)\big)\right|ds\right)$$

$$\leq L\left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right]|u\,(t)|\left(\int_0^T \gamma(s)\psi\,(|u(s)|)ds\right)$$

$$+F_0\left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right]\left(\int_0^T \gamma(s)\psi\,(|u(s)|)ds\right)$$

$$\leq L\left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right]\|\gamma\|_{L^1}\,\psi(\|u\|)|u\,(t)| + F_0\left[\frac{e^{mT}+1}{2m(e^{mT}-1)}\right]\|\gamma\|_{L^1}\,\psi(\|u\|) \tag{3.17}$$

Taking the supremum in the above inequality (3.16)

$$\|u\| \leq \frac{F_0\left[\dfrac{e^{mT}+1}{2m(e^{mT}-1)}\right]\|\gamma\|_{L^1}\,\psi(\|u\|)}{1 - L\left[\dfrac{e^{mT}+1}{2m(e^{mT}-1)}\right]\|\gamma\|_{L^1}\,\psi(\|u\|)}$$

Substituting $\|u\| = r$ in above inequality,

$$r \leq \frac{F_0\left[\dfrac{e^{mT}+1}{2m(e^{mT}-1)}\right]\|\gamma\|_{L^1}\psi(r)}{1 - L\left[\dfrac{e^{mt}+1}{2m(e^{mT}-1)}\right]\|\gamma\|_{L^1}\psi(r)}$$

This is contradiction to inequality (3.12). Hence the conclusion (ii) of theorem (2.2) does not hold. Therefore, the operator equation $AxBx = x$ and consequently the $PBVP$ (1.1) has a solution defined on J. This completes the proof.

4. An Example: Given a closed and bounded interval $J = [0,2\pi]$ of the real line $\mathbb{R}$, consider the $PBVP$ of ordinary second order differential equation,

$$-\frac{d^2}{dt^2}\left[\frac{x(t)}{a+bx(t)}\right] = -\left[\frac{x(t)}{a+bx(t)}\right] = \frac{tx^2(t)}{\pi^2[1+x^2(t)]} \qquad a.e.\ \ t \in J.$$

$$x(0) = x\,(2\pi), \qquad x'(0) = x'(2\pi) \tag{4.1}$$

where $a, b \in \mathbb{R}$ satisfying $a \neq 0$ and $a + bx > 0$ for all $x \in \mathbb{R}_+$.

Here $f(t,x) = a + bx > 0$

And $g(t,x) = -\left[\frac{x}{a+bx}\right] + \frac{tx^2}{\pi^2[1+x^2]}$

for all $t \in J$ and all $x \in \mathbb{R}_+$. Taking $m = 1$, we obtain

$$g_m(t,x) = g_1(t,x) = \frac{tx^2}{\pi^2[1+x^2]} > 0$$

for all $t \in J$ and all $x \in \mathbb{R}_+$.

It is easy to verify that all f is continuous and satisfies the hypotheses (A_0) and (A_4) in view of Remark (2.6). Further, f is Lipschitz on

$J \times \mathbb{R}$ with Lipschitz function $\ell(t) = |b|$ for $t \in J$ and so , $L = \sup_{t\in J} = |b|$.

Now,

$$|g_m(t,x)| = g_1(t,x) = \left|\frac{tx^2}{\pi^2[1+x^2]}\right| = \frac{t}{\pi^2} = \gamma(t)\psi(|x|)$$

for all $t \in J$ and all $x \in \mathbb{R}_+$.Where $\gamma(t) = \frac{t}{\pi^2}$ and $\psi(r)$ =1 for all $r > 0$.

Hence

$$\|\gamma\|_{L^1} = \frac{1}{\pi^2}\int_0^{2\pi} t\,dt = 2$$

Again,

$$L\left[\frac{e^{2m\pi}+1}{2m\,(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(r) = |b|\,\frac{(e^{2\pi}+1)}{(e^{2\pi}-1)}$$

Thus, if $|b| < \frac{e^{2\pi}-1}{(e^{2\pi}+1)}$, Then $|b|\frac{e^{2\pi}+1}{(e^{2\pi}-1)} < 1$,

And consequently, by theorem (3.11), the $PBVP$ (4.1) has a solution in a closed ball $\overline{\mathcal{B}r(0)}$, where the number r satisfies the inequality

$$r > \frac{|a|(e^{2\pi}+1)}{(e^{2\pi}-1)-|b|(e^{2\pi}+1)} \quad \text{defined on} \quad J.$$

References:

1. **J. J. Nieto,** *Nonlinear Periodic Boundary Value Problems with Caratheodory Functions, Appl. Anal.* 34 (1989), 111-128.

2. **A. Granas, R. B. Guenther and J. W. Lee,** *Some general existence principles for Caratheodory theory of nonlinear*

differential equations, J. Math. Pures et Appl. 70(1991), 153-196

3. **B. C. Dhage,** *Second order Caratheodory and discontinus differential equations in banach algebras,* Comm. Applied Analysis 10(2006), No.2, 161-175.
4. **B. C. Dhage,** *Nonlinear functional Boundary value problems in Banach Algebras Involving Caratheodories,* Kyungpook Math. J. 46(2006) 527-541.
5. **B. C. Dhage, S. N. Salunkhe, Ravi Agarwal and W. zhang,** *A functional differential equations in Banach algebras,* Math. Inequ. Appl. **8**(1) (2005), 89-99.
6. **B. C. Dhage, S. N. Salunkhe and R. U. Verma,** *A neutral functional differential equation in Banach Algebras*. Math. Sci. Resarch. J. 9(4) (2005), 101-113.
7. **Salunkhe S. N., Timol M. G. –** *Positivity Results For Functional Differential Equations on Unbounded Intervals.* VNSGU Journal of Science and Technology V1(2)- (2009), 161-168.