Optimal Bounds for Neuman-Sándor Mean in Terms of the Convex Combination of Geometric and Contra-harmonic Means

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Abstract. In this paper, we present the least value α and the greatest value β such that the double inequality

$$\alpha G(a,b) + (1-\alpha)C(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)C(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, where G(a,b), M(a,b) and C(a,b) are respectively the geometric, Neuman-Sándor and contra-harmonic means of a and b.

§1 Introduction

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a,b)$[1] is defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}, \tag{1.1}$$

where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1,2].

Let $H(a,b) = (2ab)/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\begin{aligned} &\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < P(a,b) < A(a,b) \\ &< M(a,b) < T(a,b) < Q(a,b) < C(a,b) < \max\{a,b\} \end{aligned} \tag{1.2}$$

hold for all $a, b > 0$ with $a \neq b$.

In [3], Li etc showed that the double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ holds for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(a^{p+1} - b^{p+1})/((p+1)(b-a))]^{1/p} (p \neq -1, 0)$, $L_0(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

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In [4], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b) \tag{1.3}$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \tag{1.4}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots, \beta \geq 1/3, \lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$ and $\mu \geq 1/6$.

In[5], Chu etc proved that the double inequalities

$$\alpha_1 L(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 L(a,b) + (1-\beta_1)Q(a,b) \tag{1.5}$$

and

$$\alpha_2 L(a,b) + (1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b) + (1-\beta_2)C(a,b) \tag{1.6}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \geq 2/5, \beta_1 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots, \alpha_2 \geq 5/8$ and $\beta_2 \leq 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

The main purpose of this paper is to find the least value α and the greatest value β such that the double inequality

$$\alpha G(a,b) + (1-\alpha)C(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)C(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. The using method is different from the proof method of the inequalities (1.3)-(1.6) in this paper, which has better applicability to solve such problems.

§2 Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

Lemma 2.1. *Let $f(x) = 1/\sqrt{1+x^2}$, $g(x) = 1/\sqrt{1-x^2}$ and $h(x) = \sqrt{1-x^2}$. Then the inequalities*

$$1 - \frac{x^2}{2} < f(x) < 1 - \frac{x^2}{2} + \frac{3}{8}x^4, \tag{2.1}$$

$$g(x) > 1 + \frac{x^2}{2}, \tag{2.2}$$

and

$$h(x) < 1 - \frac{x^2}{2}, \tag{2.3}$$

hold for all $x \in (0,1)$.

Proof. The first inequality in (2.1) is known (see [5, lemma 2.1]). The second inequality in (2.1) and the inequalities (2.2), (2.3) follow in turn from the inequalities

$$\left(1 - \frac{x^2}{2} + \frac{3}{8}x^4\right)^2 - f^2(x) = \frac{x^6}{64(1+x^2)}[9x^2(x^2+1) + 4(10-6x^2)] > 0, \tag{2.4}$$

$$g^2(x) - \left(1 + \frac{x^2}{2}\right)^2 = \frac{x^4}{4(1-x^2)}(x^2+3) > 0, \tag{2.5}$$

and

$$\left(1 - \frac{x^2}{2}\right)^2 - h^2(x) = \frac{x^4}{4} > 0 \tag{2.6}$$

for all $x \in (0,1)$. □

Lemma 2.2. *Let $\varphi(x) = \log(x + \sqrt{1+x^2})$. Then the inequalities*

$$x - \frac{x^3}{6} < \varphi(x) < x - \frac{x^3}{6} + \frac{3}{40}x^5, \tag{2.7}$$

hold for all $x \in (0, 1)$.

Proof. Let

$$\varphi_1(x) = \varphi(x) - (x - \frac{x^3}{6}). \tag{2.8}$$

Simple computations lead to

$$\lim_{x\to 0^+} \varphi_1(x) = 0, \tag{2.9}$$

and

$$\varphi_1'(x) = \frac{1}{\sqrt{1+x^2}} - (1 - \frac{x^2}{2}). \tag{2.10}$$

From (2.10) and the first inequality in (2.1) we know $\varphi_1'(x) > 0$. Therefore the first inequality in (2.7) follows from (2.9) and $\varphi_1'(x) > 0$.

Let

$$\varphi_2(x) = \varphi(x) - (x - \frac{x^3}{6} + \frac{3}{40}x^5). \tag{2.11}$$

Simple computations yield

$$\lim_{x\to 0^+} \varphi_2(x) = 0, \tag{2.12}$$

and

$$\varphi_2'(x) = \frac{1}{\sqrt{1+x^2}} - (1 - \frac{x^2}{2} + \frac{3}{8}x^4). \tag{2.13}$$

From (2.13) and the second inequality in (2.1) we know $\varphi_2'(x) < 0$. Therefore the second inequality in (2.7) follows from (2.12) and $\varphi_2'(x) < 0$. □

Lemma 2.3. *The inequality*

$$\sqrt{1-x^2} < \frac{4(-9x^6 + 11x^4 - 100x^2 + 150)}{5(9x^4 - 20x^2 + 120)} \tag{2.14}$$

holds for all $x \in (0, 1)$.

Proof. Let

$$\begin{aligned}\psi(x) &= [5(9x^4 - 20x^2 + 120)\sqrt{1-x^2}]^2 - [4(-9x^6 + 11x^4 - 100x^2 + 150)]^2 \\ &= -x^4\psi_1(x),\end{aligned} \tag{2.15}$$

where

$$\psi_1(x) = 1296x^8 - 1143x^6 + 19711x^4 - 5400x^2 + 28800. \tag{2.16}$$

Making use of the transform $x^2 = 1/t$ $(t \in (1, +\infty))$ for $\psi_1(x)$ yields

$$\psi_1(x) = \frac{1}{t^4}\psi_2(t), \tag{2.17}$$

where

$$\psi_2(t) = 28800t^4 - 5400t^3 + 19711t^2 - 1143t + 1296. \tag{2.18}$$

Simple computations lead to

$$\lim_{t\to 1^+} \psi_2(t) = 43264 > 0, \tag{2.19}$$

$$\psi_2'(t) = 115200t^3 - 16200t^2 + 39422t - 1143, \tag{2.20}$$

$$\lim_{t\to 1^+} \psi_2'(t) = 137279 > 0, \tag{2.21}$$

and

$$\psi_2''(t) = 2[172800t(t-1) + 156600t + 19711] > 0 \quad (2.22)$$

holds for all $t \in (1, +\infty)$. From (2.22) we clearly see that $\psi_2'(t)$ is strictly increasing in $(1, +\infty)$. It follows from (2.21), (2.19), (2.17) and (2.15) together with the monotonicity of $\psi_2'(t)$ that $\psi(x) < 0$ for $x \in (0, 1)$, this fact together with the inequalities

$$-9x^6 + 11x^4 - 100x^2 + 150 > -9 \cdot 1^6 + 11 \cdot 0^4 - 100 \cdot 1^2 + 150 = 41 > 0$$

and

$$9x^4 - 20x^2 + 120 > 9 \cdot 0^4 - 20 \cdot 1^2 + 120 = 100 > 0$$

for $x \in (0, 1)$ lead to the conclusion of lemma 2.3. □

Lemma 2.4. *Let* $\lambda = 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$ *and*

$$\begin{aligned} F(x) = \ & -4(3\lambda+2)x^{18} + 15(7\lambda-4)x^{16} + 33(7\lambda+6)x^{14} + 2(517\lambda+52)x^{12} \\ & -6(140\lambda+131)x^{10} + 3(419\lambda+216)x^8 - 5(1469\lambda-58)x^6 \\ & +24(61\lambda-27)x^4 - 6(259\lambda-51)x^2 + 4(15\lambda-11). \end{aligned} \quad (2.23)$$

Then the inequality

$$F(x) < 0 \quad (2.24)$$

holds for all $x \in (0, 1/\sqrt{3})$.

Proof. Let $x^2 = t, (t \in (0, 1/3))$, then

$$\begin{aligned} F(x) &= -4(3\lambda+2)t^9 + 15(7\lambda-4)t^8 + 33(7\lambda+6)t^7 + 2(517\lambda+52)t^6 \\ &\quad -6(140\lambda+131)t^5 + 3(419\lambda+216)t^4 - 5(1469\lambda-58)t^3 \\ &\quad +24(61\lambda-27)t^2 - 6(259\lambda-51)t + 4(15\lambda-11) \\ &= F_1(t). \end{aligned} \quad (2.25)$$

Simple computations yield

$$\lim_{t\to 0^+} F_1(t) = -4(11-15\lambda) < 0, \quad (2.26)$$

$$\begin{aligned} F_1'(t) = \ & 3[-12(3\lambda+2)t^8 + 40(7\lambda-4)t^7 + 77(7\lambda+6)t^6 + 4(517\lambda+52)t^5 \\ & -10(140\lambda+131)t^4 + 4(419\lambda+216)t^3 - 5(1469\lambda-58)t^2 \\ & +16(61\lambda-27)t - 2(259\lambda-51)], \end{aligned} \quad (2.27)$$

$$\lim_{t\to 0^+} F_1'(t) = -6(259\lambda-51) < 0, \quad (2.28)$$

$$\begin{aligned} F_1''(t) = \ & 6[-48(3\lambda+2)t^7 + 140(7\lambda-4)t^6 + 231(7\lambda+6)t^5 + 10(517\lambda+52)t^4 \\ & -20(140\lambda+131)t^3 + 6(419\lambda+216)t^2 - 5(1469\lambda-58)t + 8(61\lambda-27)], \end{aligned} \quad (2.29)$$

$$\lim_{t\to 0^+} F_1''(t) = 48(61\lambda-27) < 0, \quad (2.30)$$

$$\begin{aligned} F_1'''(t) = \ & 6[-336(3\lambda+2)t^6 + 840(7\lambda-4)t^5 + 1155(7\lambda+6)t^4 + 40(517\lambda+52)t^3 \\ & -60(140\lambda+131)t^2 + 12(419\lambda+216)t - 5(1469\lambda-58)], \end{aligned} \quad (2.31)$$

$$\lim_{t\to \frac{1}{3}^-} F_1'''(t) = \frac{64}{81}(3254 - 43389\lambda) < 0, \quad (2.32)$$

and

$$\begin{aligned} F_1^{(4)}(t) = \ & 72[-168(3\lambda+2)t^5 + 350(7\lambda-4)t^4 + 385(7\lambda+6)t^3 \\ & +10(517\lambda+52)t^2 - 10(140\lambda+131)t + (419\lambda+216)]. \end{aligned} \quad (2.33)$$

Again making use of the transform $t = 1/u, \ u \in (3, +\infty)$ for $F_1^{(4)}(t)$, we get

$$F_1^{(4)}(t) = \frac{72}{u^5} F_2(u) \quad (2.34)$$

where

$$F_2(u) = (419\lambda+216)u^5 - 10(140\lambda+131)u^4 + 10(517\lambda+52)u^3 + 385(7\lambda+6)u^2 + 350(7\lambda-4)u - 168(3\lambda+2). \quad (2.35)$$

Simple computations lead to

$$\lim_{u\to 3^+} F_2(u) = 12(13259\lambda - 1944) > 0, \quad (2.36)$$

$$F_2'(u) = 5[(419\lambda+216)u^4 - 8(140\lambda+131)u^3 + 6(517\lambda+52)u^2 + 154(7\lambda+6)u + 70(7\lambda-4)]. \quad (2.37)$$

$$\lim_{u\to 3^+} F_2'(u) = 5(35341\lambda - 5500) > 0, \quad (2.38)$$

$$F_2''(u) = 10[2(419\lambda+216)u^3 - 12(140\lambda+131)u^2 + 6(517\lambda+52)u + 77(7\lambda+6)], \quad (2.39)$$

$$\lim_{u\to 3^+} F_2''(u) = 10(17351\lambda - 1086) > 0, \quad (2.40)$$

$$F_2'''(u) = 60[(419\lambda+216)u^2 - 4(140\lambda+131)u + (517\lambda+52)], \quad (2.41)$$

$$\lim_{u\to 3^+} F_2'''(u) = 480(326\lambda + 53) > 0, \quad (2.42)$$

and

$$F_2^{(4)}(u) = 120[(419\lambda+216)(u-3) + 977\lambda + 386] > 0 \quad (2.43)$$

for $u \in (3, +\infty)$.

From(2.43) we clearly see that $F_2'''(u)$ is strictly increasing in $(3, +\infty)$. Hence $F_1^{(4)}(t) > 0$ for $t \in (0, 1/3)$ follows from (2.42), (2.40), (2.38), (2.36) and (2.34) together with the monotonicity of $F_2'''(u)$. Therefore, the conclusion of lemma 2.4 follows easily from (2.32), (2.30), (2.28), (2.26) and (2.25) together with $F_1^{(4)}(t) > 0$. □

Lemma 2.5. *Let* $\lambda = 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$ *and*

$$H(x) = -20\lambda x^{18} + (53\lambda-8)x^{16} + 3(155\lambda-12)x^{14} + 2(400\lambda+169)x^{12} - 6(154\lambda+117)x^{10} + 3(507\lambda+128)x^8 - (7487\lambda-432)x^6 + 2(743\lambda-335)x^4 - 6(259\lambda-51)x^2 + 4(15\lambda-11). \quad (2.44)$$

Then the inequality

$$H(x) < 0 \quad (2.45)$$

holds for all $x \in (1/2, 1)$.

Proof. Let $x^2 = t$ ($t \in (1/4, 1)$). Then

$$\begin{aligned} H(x) &= -20\lambda t^9 + (53\lambda-8)t^8 + 3(155\lambda-12)t^7 + 2(400\lambda+169)t^6 \\ &\quad -6(154\lambda+117)t^5 + 3(507\lambda+128)t^4 - (7487\lambda-432)t^3 \\ &\quad +2(743\lambda-335)t^2 - 6(259\lambda-51)t + 4(15\lambda-11) \\ &= H_1(t). \end{aligned} \quad (2.46)$$

Simple computations yield

$$\lim_{t\to \frac{1}{4}^+} H_1(t) = -\frac{5}{16384}(1138183\lambda + 5670) < 0, \quad (2.47)$$

$$H_1'(t) = -180\lambda t^8 + 8(53\lambda-8)t^7 + 21(155\lambda-12)t^6 + 12(400\lambda+169)t^5 - 30(154\lambda+117)t^4 + 12(507\lambda+128)t^3 - 3(7487\lambda-432)t^2 + 4(743\lambda-335)t - 6(259\lambda-51), \quad (2.48)$$

$$\lim_{t\to \frac{1}{4}^+} H_1'(t) = -\frac{15}{16384}(2329031\lambda - 70128) < 0, \quad \lim_{t\to 1^-} H_1'(t) = -11280\lambda < 0, \quad (2.49)$$

$$H_1''(t) = 2[-720\lambda t^7 + 28(53\lambda-8)t^6 + 63(155\lambda-12)t^5 + 30(400\lambda+169)t^4 - 60\cdot (154\lambda+117)t^3 + 18(507\lambda+128)t^2 - 3(7487\lambda-432)t + 2(743\lambda-335)], \quad (2.50)$$

$$\lim_{t\to\frac{1}{4}^+} H_1''(t) = -\frac{9}{512}(414893\lambda+33300) < 0, \ \lim_{t\to 1^-} H_1''(t) = 2880\lambda > 0, \quad (2.51)$$

$$H_1'''(t) = 6[-1680\lambda t^6 + 56(53\lambda-8)t^5 + 105(155\lambda-12)t^4 + 40(400\lambda+169)t^3 -60(154\lambda+117)t^2 + 12(507\lambda+128)t - (7487\lambda-432)], \quad (2.52)$$

$$\lim_{t\to\frac{1}{4}^+} H_1'''(t) = -\frac{9}{32}(132852\lambda-10187) < 0, \ \lim_{t\to 1^-} H_1'''(t) = 137520\lambda > 0, \quad (2.53)$$

$$H_1^{(4)}(t) = 24[-2520\lambda t^5 + 70(53\lambda-8)t^4 + 105(155\lambda-12)t^3 +30(400\lambda+169)t^2 - 30(154\lambda+117)t + 3(507\lambda+128)], \quad (2.54)$$

$$\lim_{t\to\frac{1}{4}^+} H_1^{(4)}(t) = \frac{3}{8}(88469\lambda-12704) > 0, \quad (2.55)$$

$$H_1^{(5)}(t) = 120[-2520\lambda t^4 + 56(53\lambda-8)t^3 + 63(155\lambda-12)t^2 +12(400\lambda+169)t - 6(154\lambda+117)], \quad (2.56)$$

$$\lim_{t\to\frac{1}{4}^+} H_1^{(5)}(t) = \frac{15}{4}(29531\lambda-7976) > 0, \quad (2.57)$$

$$H_1^{(6)}(t) = 720[-1680\lambda t^3 + 28(53\lambda-8)t^2 + 21(155\lambda-12)t + 2(400\lambda+169)], \quad (2.58)$$

$$\lim_{t\to\frac{1}{4}^+} H_1^{(6)}(t) = 180(6721\lambda+1044) > 0, \ \lim_{t\to 1^-} H_1^{(6)}(t) = 720(3859\lambda-138) > 0, \quad (2.59)$$

$$H_1^{(7)}(t) = 5040[-720\lambda t^2 + 8(53\lambda-8)t + 3(155\lambda-12)], \quad (2.60)$$

and

$$H_1^{(8)}(t) = 40320[-180\lambda t + (53\lambda-8)] < 322560(\lambda-1) < 0, \quad (2.61)$$

for all $t \in (1/4, 1)$.

From (2.61) we clearly see that $H_1^{(6)}(t)$ is concave function in $(1/4, 1)$. It follows from (2.59) together with the concavity of $H_1^{(6)}(t)$ that $H_1^{(6)}(t) > 0$ for $t \in (1/4, 1)$, hence $H_1^{(5)}(t)$ is strictly increasing in $(1/4, 1)$.

From (2.55) and (2.57) together with the monotonicity of $H_1^{(5)}(t)$ we know that $H_1^{(4)}(t) > 0$ for $t \in (1/4, 1)$, thus $H_1'''(t)$ is strictly increasing in $(1/4, 1)$. It follows from (2.53) and the monotonicity of $H_1'''(t)$ that there exists $\mu_1 \in (1/4, 1)$ such that $H_1'''(t) < 0$ for $t \in (1/4, \mu_1)$ and $H_1'''(t) > 0$ for $t \in (\mu_1, 1)$, hence $H_1''(t)$ is strictly decreasing in $(1/4, \mu_1)$ and strictly increasing in $(\mu_1, 1)$.

From (2.51) and the monotonicity of $H_1''(t)$ we affirm that there exists $\mu_2 \in (1/4, 1)$ such that $H_1''(t) < 0$ for $t \in (1/4, \mu_2)$ and $H_1''(t) > 0$ for $t \in (\mu_2, 1)$, thus $H_1'(t)$ is strictly decreasing in $(1/4, \mu_2)$ and strictly increasing in $(\mu_2, 1)$. It follows from (2.49) and the monotonicity of $H_1'(t)$ that $H_1'(t) < 0$ for $t \in (1/4, 1)$, hence $H_1(t)$ is strictly decreasing in (1/4,1).

Therefore, the conclusion of lemma 2.5 follows from (2.47) and (2.46) together with the monotonicity of $H_1(t)$.

□

§3 Main Results

Theorem 3.1. *The double inequality*

$$\alpha G(a,b) + (1-\alpha)C(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)C(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \geq 5/9$ *and* $\beta \leq 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$ and $\lambda = 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$. Then

$$\frac{G(a,b)}{A(a,b)} = \sqrt{1-x^2}, \frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}(x)}, \frac{C(a,b)}{A(a,b)} = 1 + x^2. \tag{3.1}$$

Firstly, we prove that

$$\frac{5}{9}G(a,b) + \frac{4}{9}C(a,b) < M(a,b). \tag{3.2}$$

Equations (3.1) lead to

$$\begin{aligned} \frac{5G(a,b)}{9A(a,b)} + \frac{4C(a,b)}{9A(a,b)} - \frac{M(a,b)}{A(a,b)} &= \frac{5}{9}\sqrt{1-x^2} + \frac{4}{9}(1+x^2) - \frac{x}{\sinh^{-1}(x)} \\ &= \frac{1}{\log(x+\sqrt{1+x^2})}d(x), \end{aligned} \tag{3.3}$$

where

$$d(x) = [\frac{5}{9}\sqrt{1-x^2} + \frac{4}{9}(1+x^2)]\log(x+\sqrt{1+x^2}) - x. \tag{3.4}$$

Making use of the conclusion of lemmas 2.2 and 2.3 for (3.4) yields

$$d(x) < \left[\frac{4(-9x^6+11x^4-100x^2+150)}{9(9x^4-20x^2+120)} + \frac{4(1+x^2)}{9}\right](x - \frac{x^3}{6} + \frac{3x^5}{40}) - x = 0. \tag{3.5}$$

Therefore, inequality (3.2) follows from (3.3) and (3.5).

Secondly, we prove that

$$\lambda G(a,b) + (1-\lambda)C(a,b) > M(a,b) \tag{3.6}$$

Equations (3.1) lead to

$$\begin{aligned} &\frac{\lambda G(a,b)}{A(a,b)} + \frac{(1-\lambda)C(a,b)}{A(a,b)} - \frac{M(a,b)}{A(a,b)} \\ = &\ \lambda\sqrt{1-x^2} + (1-\lambda)(1+x^2) - \frac{x}{\log(x+\sqrt{1+x^2})} \\ = &\ \frac{D(x)}{\log(x+\sqrt{1+x^2})}, \end{aligned} \tag{3.7}$$

where

$$D(x) = [\lambda\sqrt{1-x^2} + (1-\lambda)(1+x^2)]\log(x+\sqrt{1+x^2}) - x. \tag{3.8}$$

Some tedious, but not difficult, computations lead to

$$\lim_{x\to 0^+} D(x) = 0, \tag{3.9}$$

$$\lim_{x\to 1^-} D(x) = 0, \tag{3.10}$$

$$\begin{aligned} D'(x) = &\ x\left[2(1-\lambda) - \frac{\lambda}{\sqrt{1-x^2}}\right]\log(x+\sqrt{1+x^2}) \\ &+\lambda\sqrt{\frac{1-x^2}{1+x^2}} + (1-\lambda)\sqrt{1+x^2} - 1, \end{aligned} \tag{3.11}$$

$$\lim_{x\to 0^+} D'(x) = 0, \tag{3.12}$$

$$\lim_{x\to 1^-} D'(x) = -\infty, \tag{3.13}$$

$$\begin{aligned} D''(x) = & \left[2(1-\lambda) - \frac{\lambda}{(1-x^2)^{\frac{3}{2}}}\right] \log(x+\sqrt{1+x^2}) \\ & -\frac{\lambda x(3+x^2)}{(1+x^2)\sqrt{1-x^4}} + \frac{3(1-\lambda)x}{\sqrt{1+x^2}}, \end{aligned} \tag{3.14}$$

$$\lim_{x\to 0^+} D''(x) = 0, \tag{3.15}$$

$$\lim_{x\to 1^-} D''(x) = -\infty, \tag{3.16}$$

$$\begin{aligned} D'''(x) = & -\frac{3\lambda x}{(1-x^2)^{\frac{5}{2}}} \log(x+\sqrt{1+x^2}) \\ & -\frac{\lambda(x^6+8x^4-x^2+4)}{(1+x^2)(1-x^4)^{\frac{3}{2}}} + \frac{(1-\lambda)(2x^2+5)}{(1+x^2)^{\frac{3}{2}}}, \end{aligned} \tag{3.17}$$

$$\lim_{x\to 0^+} D'''(x) = 5-9\lambda > 0, \tag{3.18}$$

$$\lim_{x\to 1^-} D'''(x) = -\infty, \tag{3.19}$$

$$\begin{aligned} D^{(4)}(x) = & -\frac{3\lambda(4x^2+1)}{(1-x^2)^{\frac{7}{2}}} \log(x+\sqrt{1+x^2}) - \frac{(1-\lambda)x(2x^2+11)}{(1+x^2)^{\frac{5}{2}}} \\ & -\frac{\lambda x(2x^8+33x^6-7x^4+75x^2-7)}{(1+x^2)(1-x^4)^{\frac{5}{2}}}, \end{aligned} \tag{3.20}$$

$$\lim_{x\to 0^+} D^{(4)}(x) = 0, \tag{3.21}$$

and

$$\begin{aligned} D^{(5)}(x) = & -\frac{15\lambda x(4x^2+3)}{(1-x^2)^{\frac{9}{2}}} \log(x+\sqrt{1+x^2}) + \frac{(1-\lambda)(4x^4+38x^2-11)}{(1+x^2)^{\frac{7}{2}}} \\ & -\frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{(1+x^2)(1-x^4)^{\frac{7}{2}}}. \end{aligned} \tag{3.22}$$

Making use of the first inequality of (2.7) for (3.22) yields

$$\begin{aligned} D^{(5)}(x) < & -\frac{15\lambda x(4x^2+3)}{(1-x^2)^{\frac{9}{2}}}(x-\frac{x^3}{6}) + \frac{(1-\lambda)(4x^4+38x^2-11)}{(1+x^2)^{\frac{7}{2}}} \\ & -\frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{(1+x^2)(1-x^4)^{\frac{7}{2}}} \\ = & \frac{1}{(1-x^4)^{\frac{7}{2}}} D_1(x), \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} D_1(x) = & -\frac{5\lambda x(4x^2+3)(6x-x^3)}{2(1-x^2)}(1+x^2)^{\frac{7}{2}} + (1-\lambda)(4x^4+38x^2-11)(1-x^2)^{\frac{7}{2}} \\ & -\frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{1+x^2}. \end{aligned} \tag{3.24}$$

Factoring for $4x^4+38x^2-11$ in (3.24) leads to

$$\begin{aligned} D_1(x) = & (1-\lambda)(4x^2+9\sqrt{5}-19)\left(x+\frac{1}{2}\sqrt{9\sqrt{5}-19}\right)\left(x-\frac{1}{2}\sqrt{9\sqrt{5}-19}\right)\cdot \\ & (1-x^2)^{\frac{7}{2}} - \frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{1+x^2} \\ & -\frac{5\lambda x(4x^2+3)(6x-x^3)}{2(1-x^2)}(1+x^2)^{\frac{7}{2}}. \end{aligned} \tag{3.25}$$

In order to discuss $D_1(x)$ is positive or negative, we divide the range of variable x into two intervals $\left(0, 1/2\sqrt{9\sqrt{5}-19}\right]$ and $(1/2\sqrt{9\sqrt{5}-19}, 1)$.

For $x \in \left(0, 1/2\sqrt{9\sqrt{5}-19}\right]$, (3.25) is rewritten into

$$\begin{aligned} D_1(x) = \; & (1-\lambda)(4x^2+9\sqrt{5}-19)\left(x+\frac{1}{2}\sqrt{9\sqrt{5}-19}\right)\left(x-\frac{1}{2}\sqrt{9\sqrt{5}-19}\right)\cdot \\ & (1-x^2)^4 g(x) - \frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{1+x^2} \\ & -\frac{5\lambda x(4x^2+3)(6x-x^3)}{2(1-x^2)}(1+x^2)^4 f(x). \end{aligned} \tag{3.26}$$

where $f(x)$ and $g(x)$ are defined as in lemma 2.1. Making use of the first inequality of (2.1) and the inequality (2.2) for (3.26), and noticing the interval $\left(0, 1/2\sqrt{9\sqrt{5}-19}\right] \subset (0, 1/\sqrt{3})$ we have

$$\begin{aligned} D_1(x) \; < \; & (1-\lambda)(4x^4+38x^2-11)(1-x^2)^4(1+\frac{x^2}{2}) \\ & -\frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{1+x^2} \\ & -\frac{5\lambda x(4x^2+3)(6x-x^3)}{2(1-x^2)}(1+x^2)^4(1-\frac{x^2}{2}) \\ = \; & \frac{1}{4(1-x^4)}F(x). \end{aligned} \tag{3.27}$$

where $F(x)$ is defined as in lemma 2.4. It fllows from (3.27) and lemma 2.4 that

$$D_1(x) < 0. \tag{3.28}$$

For $x \in (1/2\sqrt{9\sqrt{5}-19}, 1)$, (3.25) is rewritten into

$$\begin{aligned} D_1(x) = \; & (1-\lambda)(4x^2+9\sqrt{5}-19)\left(x+\frac{1}{2}\sqrt{9\sqrt{5}-19}\right)\left(x-\frac{1}{2}\sqrt{9\sqrt{5}-19}\right)\cdot \\ & (1-x^2)^3 h(x) - \frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{1+x^2} \\ & -\frac{5\lambda x(4x^2+3)(6x-x^3)}{2(1-x^2)}(1+x^2)^4 f(x). \end{aligned} \tag{3.29}$$

where $f(x), h(x)$ are defined as in lemma 2.1. Making use of the first inequality of (2.1) and the inequality (2.3) for (3.29), and noticing the interval $(1/2\sqrt{9\sqrt{5}-19}, 1) \subset (1/2, 1)$ one has

$$\begin{aligned} D_1(x) \; < \; & (1-\lambda)(4x^4+38x^2-11)(1-x^2)^3(1-\frac{x^2}{2}) \\ & -\frac{\lambda(6x^{12}+173x^{10}-46x^8+1004x^6-196x^4+263x^2-4)}{1+x^2} \\ & -\frac{5\lambda x(4x^2+3)(6x-x^3)}{2(1-x^2)}(1+x^2)^4(1-\frac{x^2}{2}) \\ = \; & \frac{1}{4(1-x^4)}H(x). \end{aligned} \tag{3.30}$$

where $H(x)$ is defined as in lemma 2.5. It fllows from (3.30) and lemma 2.5 that

$$D_1(x) < 0. \tag{3.31}$$

Synthesizing the above two cases we affirm that $D_1(x) < 0$ for all $x \in (0, 1)$. This fact and (3.23) imply that $D^{(5)}(x) < 0$ for all $x \in (0, 1)$, thus $D^{(4)}(x)$ is strictly decreasing in (0,1).

From (3.21) and the monotonicity of $D^{(4)}(x)$ we know that $D^{(4)}(x) < 0$ for $x \in (0, 1)$, hence $D'''(x)$ is strictly decreasing in (0,1). It follows from (3.18) and (3.19) together with the monotonicity of $D'''(x)$ that there exists $\nu_1 \in (0,1)$ such that $D'''(x) > 0$ for $x \in (0, \nu_1)$ and $D'''(x) < 0$ for $x \in (\nu_1, 1)$, hence $D''(x)$ is strictly increasing in $(0, \nu_1)$ and strictly decreasing in $(\nu_1, 1)$.

From (3.15) and (3.16) together with the monotonicity of $D''(x)$ we clearly see that there exists $\nu_2 \in (0,1)$ such that $D''(x) > 0$ for $x \in (0, \nu_2)$ and $D''(x) < 0$ for $x \in (\nu_2, 1)$, hence $D'(x)$ is strictly increasing in $(0, \nu_2)$ and strictly decreasing in $(\nu_2, 1)$.

From (3.12) and (3.13) together with the monotonicity of $D'(x)$ know that there exists $\nu_3 \in (0,1)$ such that $D'(x) > 0$ for $x \in (0, \nu_3)$ and $D'(x) < 0$ for $x \in (\nu_3, 1)$, hence $D(x)$ is strictly increasing in $(0, \nu_3)$ and strictly decreasing in $(\nu_3, 1)$. It follows from (3.9) and (3.10) together with the monotonicity of $D(x)$ that

$$D(x) > 0 \tag{3.32}$$

for all $x \in (0,1)$.

Therefore the inequality (3.6) follows from (3.7) and (3.32).

At least, we prove that $5/9G(a,b) + 4/9C(a,b)$ is the best possible lower convex combination bound and $\lambda G(a,b) + (1-\lambda)C(a,b)$ is the best possible upper convex combination bound of the geometric and contra-harmonic means for the Neuman-Sándor mean.

From equations (3.1) one has

$$\frac{C(a,b) - M(a,b)}{C(a,b) - G(a,b)} = \frac{(1+x^2)\log(x + \sqrt{1+x^2}) - x}{(1 + x^2 - \sqrt{1-x^2})\log(x + \sqrt{1+x^2})} = B(x). \tag{3.33}$$

It is easy to calculate that

$$\lim_{x \to 0^+} B(x) = \frac{5}{9} \tag{3.34}$$

and

$$\lim_{x \to 1^-} B(x) = \lambda. \tag{3.35}$$

If $\alpha < 5/9$, then equations (3.33) and (3.34) lead to conclusion that there exists $\delta_1 = \delta_1(\alpha) \in (0,1)$ such that $M(a,b) < \alpha G(a,b) + (1-\alpha)C(a,b)$ for $(a-b)/(a+b) \in (0, \delta_1)$.

If $\beta > \lambda$, then equations (3.33) and (3.35) imply the conclusion that there exists $\delta_2 = \delta_2(\beta) \in (0,1)$ such that $M(a,b) > \beta G(a,b) + (1-\beta)C(a,b)$ for $(a-b)/(a+b) \in (1-\delta_2, 1)$. ☐

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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