Note on THE l-TRANSLATIVITY OF Matrix Based on Convergent Infinite Geometric Series

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Abstract: The infinite Geometric Series is a series of the form $\sum_{k=0}^{\infty} ax^k$. The geometric power series $\sum_{k=0}^{\infty} ax^k$ converges for $|x|<1$ and is equal to $\frac{a}{1-x}$.

Let g be sequence in (0, 1) that converges to 1. The matrix based on convergent infinite geometric series defined as $a_{nk}= (1-g_n)g_n{}^k$. We denote this matrix by M_g and name it geometric matrix. M_g is a sequence to sequence mapping. When a matrix M_g is applied to a sequence x, we get a new sequence $M_g x$ whose nth term is given by:

$$(M_g x)_n = (1-g_n)\sum_{k=o}^{\infty} g_n{}^k x_k$$

The sequence $M_g x$ is called the M_g-transform of the sequence x.

The M_g matrix was introduced by Madison Hankson, Tiffany Northcut and Mulatu Lemma in (1).

1. Basic notation and definitions. Let $A=(a_{nk})$ be an infinite matrix defining a sequence to a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

(2.1)

Where $(Ax)_n$ denotes the nth term of the image sequence Ax. Let y be a complex number sequence. Throughout this paper, we use the following basic notations and definitions:

i. $c = \{\text{the set of all convergent complex number sequences}\}$

ii. $\ell = \{y : \sum_{k=0}^{\infty} |y_k| \, converges\}$

iii. $\ell(A) = \{y : Ay \in \ell\}$

iv. $c(A) = \{\text{y} : y \text{ is summable by A}\}$

Definition 1. If X and Y are sets of complex number sequences, then the matrix A is called an $X-Y$ matrix if the image Au of u under the transformation A is in Y whenever u is in X.

Definition 2. The summability matrix A *is* said to be l-translative for the sequence u in $\ell(A)$ provided that each of the sequences Tu and Su is in $\ell(A)$, where $T_u = \{u_1, u_2, u_3, \ldots\}$ and $S_u = \{0, u_0, u_1, \ldots\}$.

2. The main results

Proposition 1.

$$M_g \, is \;\; \ell - \ell \Leftrightarrow (1-g) \in \ell$$

Lemma 1:

$$M_g \, is \;\; \ell - \ell \Rightarrow (1-g) \in \ell$$

Proof: We use the Knopp-Lorentz Rule:

$$M_g \, is \;\; \ell - \ell \Rightarrow$$

$$\sum_{n=0}^{\infty} \left| (1-g_n) g_n^k \right| \le M$$

$$\sum_{n=0}^{\infty}\left|(1-g_n)\right| \le M \quad \text{(for k=0)}$$

$$\Rightarrow (1-g) \in \ell$$

Lemma 2:

$$1-g \in \ell \Rightarrow M_g is l-l$$

Proof: We use the Knopp-Lorentz Rule:

$$\sum_{n=0}^{\infty}\left|a_{nk}\right| \le \sum_{n=0}^{\infty}\left|(1-g_n)g_n^k\right|$$

$$\le \sum_{n=0}^{\infty}(1-g_n) \le M \quad \text{for some M>0 as}$$

$$(1-g) \in \ell .$$

Now Proposition 1 follows by Lemmas 1&2.

Proposition 2. *Every* l-l G_g *matrix is* l-translative *for each sequence* $x \in \ell$.

Theorem 1. *Every* l-l G_g *matrix is is* l-translative *for those sequences for which* $x \in l(G_g)$, k=1,2,3,4…...

Proof. Suppose that x is a sequence in $l(G_g)$. We show that

(1) $T_x \in l(G_g)$, and

(2) $S_x \in l(G_g)$, where Tx and Sx are as defined in Definition 2.

Let us first show that (1) holds.
Note that

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$$\left|(M_g T_x)_n\right| = (1-t_n)\left|\sum_{k=0}^{\infty} x_{k+1} t_n^k\right|$$

$$= \frac{(1-t_n)}{t_n}\left|\sum_{k=0}^{\infty} x_{k+1} t_n^{k+1}\right|$$

$$= \frac{(1-t_n)}{t_n}\left|\sum_{k=1}^{\infty} x_k t_n^k\right|$$

Let $$A_n = \frac{(1-t_n)}{t_n}\left|\sum_{k=1}^{\infty} x_k t_n^k\right|$$

Now the conditions that $A \in l$ follows from $x \in l(M_g)$.
Next, we show that (2) holds as follows. We have

$$\left|(M_g S_x)_n\right| = (1-t_n)\left|\sum_{k=1}^{\infty} x_{k-1} t_n^k\right|$$

$$= (1-t_n)\left|\sum_{k=0}^{\infty} x_k t_n^{k+1}\right|$$

Let $$E_n = t_n(1-t_n)\left|\sum_{k=0}^{\infty} x_k t_n^k\right|$$

But the hypothesis that $x \in l(G_g)$ implies that E *is* in ℓ. Hence the theorem follows.

Here, we remark that a sequence x defined by $x_k = (-1)^k$ is one of the sequences which satisfies the condition of Theorem 1.

Corollary 1. E*very* $l - l$ g *matrix is l-translative for the class of all sequence* x *whose partial sum is bounded.*

Proof. By [3, Thm. 8], x is in $\ell(G_g)$.Hence the assertion follows by Theorem 1.

Corollary 2. E*very* $\ell - \ell$ G_g *matrix is l*-translative *for the unbounded sequence* x *defined by* $x_k = (-1)^k (k+1)$

Proof.

.Note that

$$(M_g x)_n = \sum_{k=0}^{\infty} (1-g_n) g_n^k (-1)^k (k+1)$$

$$= (1-g_n) \sum_{k=0}^{\infty} g_n^k (-1)^k (k+1)$$

$$= (1-g_n) \sum_{k=0}^{\infty} (-g_n)^k (k+1)$$

$$= \frac{1-g_n}{(1+g_n)^2}$$

$$\leq (1-g_n)$$

Now M_g matrix is l-*l* $\Rightarrow$ *(1-g)* $\in$ l , by Proportion 1 and hence $M_g x \in l.$

Now since $x \in \ell(G_g)$, the corollary easily follows by Theorem 1.

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