Quotient Frames and Filters

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Abstract

Frame theory is the study of topology based on its open set lattice and it was studied extensively by various authors. In this paper we introduce the notion of quotient frames using filters and study the relation between the filters of the given frame and the filters of the corresponding quotient frame.

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1. Introduction

Frame theory is topology seen through notions of lattice theory; here one takes the lattice of open sets as the basic notion. The concept of Frames has been studied by many mathematicians including B. Banaschewski, C.H Dowker, P.T. Johnstone. For details one can refer to [1,2,3,4]. Here by using the concept of filters quotient frame is introduced. In this paper we introduce the notion of equivalence relation on F and the notion of quotient frame and we study the properties of the filters of the quotient.

2. Preliminaries

In this section we shall review some fundamental definitions and notions from [1,2,3,4].

Definition 2.1. A frame is a complete lattice F satisfying the infinite distributive law $a \wedge (\bigvee S) = \bigvee\{a \wedge x \mid x \in S\}$ for any $a \in F$ and $S \subseteq F$.

In particular the following are examples of a frame:

1. The open sets of a topological space ordered by set inclusion
2. Any finite distributive lattice
3. The interval I= [0,1] of **R**
4. Any complete totally ordered set.

Definition 2.2. A subset M of a frame F closed under finite meets, arbitrary joins and having unit(top) element e_F and zero(bottom) element o_F of the frame F is called a subframe of F.

Definition 2.3. For frames F, M a map $h: F \to M$ is a frame homomorphism if h preserves finite meets, arbitrary joins, unit element and zero element.

Definition 2.4. Let F be a frame. For any non empty subset P of F, P is a filter or upset of F if and only if i) $a, b \in P$ implies $a \wedge b \in P$, ii) $a \in P, x \in F$ implies $a \vee x \in P$.

3. Quotient Frame

Let P be a filter of the frame F. For any $x, y \in F$ we define a relation $\sim$ on F by $x \sim y$ if and only if there exist $a, b \in P$ such that $x \wedge a = y \wedge b$.

Proposition 3.1. The relation $\sim$ is an equivalence relation on F.

Proof. Since $e_F \in P$ we have $x \sim x$ for every $x \in F$, hence $\sim$ is reflexive. By the definition of the relation, $\sim$ is symmetric. If $x \sim y$ and $y \sim z$ then there are a, b, c and d in P such that $x \wedge a = y \wedge b$ and $y \wedge c = z \wedge d$. Hence $(x \wedge a) \wedge c = (y \wedge b) \wedge c = y \wedge (b \wedge c) = y \wedge (c \wedge b)$ $= (y \wedge c) \wedge b = (y \wedge d) \wedge b$. Thus $x \wedge (a \wedge c) = z \wedge (d \wedge b)$ and $x \sim z$. Hence $\sim$ is transitive. Therefore $\sim$ is an equivalence relation on F.

Lemma 3.2. If $x_1 \sim x_2$ then $x_1 \wedge x \sim x_2 \wedge x$ and $x_1 \vee x \sim x_2 \vee x$ for all $x \in F$.

Proof. Since $x_1 \sim x_2$, there exist $a, b \in P$ such that $x_1 \wedge a = x_2 \wedge b$. Then for all $x \in F$, $(x_1 \wedge a) \wedge x = (x_2 \wedge b) \wedge x$. Hence $(x_1 \wedge x) \wedge a = (x_2 \wedge x) \wedge b$ and $x_1 \wedge x \sim x_2 \wedge x$.

Similarly $(x_1 \wedge a) \vee x = (x_2 \wedge b) \vee x$ and so $(x_1 \vee x) \wedge (a \vee x) = (x_2 \vee x) \wedge (b \vee x)$.

Since P is filter of F, $a \vee x, b \vee x \in P$ and $x_1 \vee x \sim x_2 \vee x$.

We denote $[x]_P = \{y \in F \mid x \sim y\}$ the equivalence class of x determined by the filter P.

Proposition 3.3. Equivalent classes of F determined by the filter P satisfies

1. For each $x \in F$, $x \in [x]_P$
2. $[x]_P = [y]_P \Leftrightarrow x \sim y$
3. $x \in [y]_P \Leftrightarrow [x]_P = [y]_P \Leftrightarrow y \in [x]_P$
4. For any $x, y \in F$ exactly one of the following holds

   i) $[x]_P \cap [y]_P = \varnothing$ ii) $[x]_P = [y]_P$

Proof. 1. Since $x \sim x$ we have $x \in [x]_P$

2. $[x]_P = [y]_P \Rightarrow y \in [x]_P \Rightarrow x \sim y$.Conversely $x \sim y \Leftrightarrow y \sim x$, as $\sim$ is symmetric. Also for any $x_1 \in [x]_P$, we have $x \sim x_1$. Hence $y \sim x_1$. Thus $x_1 \in [x]_P \Rightarrow x_1 \in [y]_P$. Hence $[x]_P \subseteq [y]_P$. Similarly $[y]_P \subseteq [x]_P$. Therefore $[x]_P = [y]_P$.

3. Follows from the proof of 2.

4. For any z, $z \in [x]_P \cap [y]_P \Rightarrow x \sim z$ and $y \sim z \Rightarrow x \sim z$ and $z \sim y \Rightarrow x \sim y \Rightarrow y \in [x]_P$

Hence $[x]_P = [y]_P$ from 3. Therefore the result follows.

Lemma 3.4. For any filter *P* of *F*, $P = [e_F]_P$.

Proof. If $a \in P$ then $a \wedge e_F = e_F \wedge a$. Thus $a \sim e_F$ and $a \in [e_F]_P$. Hence $P \subseteq [e_F]_P$.

Conversely if $a \in [e_F]_P$, then $e_F \sim a$ and hence there exist $b, c \in P$ such that $e_F \wedge b = a \wedge c$.

Thus $b = a \wedge c$. Now as *P* is a filter of *F* and $a \wedge c \in P$, we have $a \in P$ as $(a \wedge c) \vee a = a$.

Hence $[e_F]_P \subseteq P$. Thus $P = [e_F]_P$.

Denote by $F/P = \{[x]_P \mid x \in F\}$ the set of all equivalence classes $[x]_P$ determined by the filter *F*. We define two operations $\tilde{\wedge}$ and $\tilde{\bigvee}$ on F/P as follows:

$$[x]_P \,\tilde{\wedge}\, [y]_P = [x \wedge y]_P \text{ and } \tilde{\bigvee_{i \in \wedge}} [x_i]_P = [\bigvee_{i \in \wedge} x_i]_P \text{ for } x, y, (x_i)_{i \in \wedge} \in F.$$

Since $\sim$ is an equivalence relation on *F*, the operations $\tilde{\wedge}$ and $\tilde{\bigvee}$ are well defined. For if $[x]_P = [x']_P$, $[y]_P = [y']_P$ and $[x_i]_P = [x_i']_P$ for $i \in \wedge$. Then $x \sim x'$, $y \sim y'$ and $x_i \sim x_i'$ for $i \in \wedge$.

Now $x_i \sim x_i' \Rightarrow x_i \wedge a = x_i' \wedge b$ for some $a, b \in P \Rightarrow \bigvee_{i \in \wedge} (x_i \wedge a) = \bigvee_{i \in \wedge} (x_i' \wedge b)$

$\Rightarrow a \wedge (\bigvee_{i \in \wedge} x_i) = b \wedge (\bigvee_{i \in \wedge} x_i')$. Hence $\bigvee_{i \in \wedge} x_i \sim \bigvee_{i \in \wedge} x_i'$. Similarly we have $x \wedge y \sim x' \wedge y'$.

Hence $[x]_P \tilde{\wedge} [y]_P = [x \wedge y]_P = [x' \wedge y']_P = [x']_P \tilde{\wedge} [y']_P$ and $\bigvee_{i\in\wedge}[x_i]_P = [\bigvee_{i\in\wedge} x_i]_P = [\bigvee_{i\in\wedge} x_i']_P = \bigvee_{i\in\wedge}[x_i']_P$.

Therefore the operation $\tilde{\wedge}$ and $\tilde{\bigvee}$ are well defined.

Theorem 3.5. Let P be a filter of the frame F. Then $(F/P, \tilde{\wedge}, \tilde{\bigvee})$ is also a frame with unity element $[e_F]_P = P$ and zero element $[o_F]_P$.

Proof: Let $[x]_P, [x_i]_P \in F/P$ for all x, $(x_i)_{i\in\wedge} \in F$.

Then $[x_i]_P \tilde{\wedge} [x_j]_P = [x_i \wedge x_j]_P = [x_j \wedge x_i]_P = [x_j]_P \tilde{\wedge} [x_i]_P$ and

$[x_i]_P \tilde{\vee} [x_j]_P = [x_i \vee x_j]_P = [x_j \vee x_i]_P = [x_j]_P \tilde{\vee} [x_i]_P$.

Also $[x]_P \tilde{\wedge} (\tilde{\bigvee}_{i\in\wedge} [x_i]_P) = [x]_P \tilde{\wedge} [\bigvee_{i\in\wedge} x_i]_P = [x \wedge \bigvee_{i\in\wedge} x_i]_P = [\bigvee_{i\in\wedge}(x \wedge x_i)]_P = \tilde{\bigvee}_{i\in\wedge}[x \wedge x_i]_P$

There exist $[e_F]_P = P$ and $[o_F]_P$ in F/P such that $[x]_P \tilde{\wedge} [e_F]_P = [x \wedge e_F]_P = [x]_P$ and

$[x]_P \tilde{\vee} [o_F]_P = [x \vee o_F]_P = [x]_P$ for all $[x]_P \in F/P$. Therefore $(F/P, \tilde{\wedge}, \tilde{\bigvee})$ is a frame.

Definition 3.6. The frame F/P described in above theorem is called the quotient frame of the frame F by the filter P.

Theorem 3.7. Let P be a filter of the frame F then,

1. $k : F \to F/P$ given by $k(x) = [x]_P$ is frame homomorphism such that $k(x) = k(y)$ whenever $x \sim y$

2. If $f : F \to F'$ is a frame homomorphism such that $x \sim y \Rightarrow f(x) = f(y)$ then there is exactly one frame homomorphism $g : F/P \to F'$ such that $g \circ k = f$

Proof: 1. We have $k(x \wedge y) = [x \wedge y]_P = [x]_P \tilde{\wedge} [y]_P$ and $k(\bigvee_{i\in\wedge} x_i) = [\bigvee_{i\in\wedge} x_i]_P = \tilde{\bigvee}_{i\in\wedge}[x_i]_P$. Again since $k(e_F) = [e_F]_P$ and , $k(o_F) = [o_F]_P$, k preserves unit and zero elements.

Also $x \sim y \Rightarrow [x]_P = [y]_P \Rightarrow k(x) = k(y)$.

2.

$$\begin{tikzcd} F \arrow[rr] \arrow[dr, "k"'] & & F' \\ & F/P \arrow[ur, "g"'] & \end{tikzcd}$$

Consider $[x]_P \in F/P$ and set $g([x]_P) = f(x)$. Then $(g \circ k)(x) = g(k(x)) = g([x]_P) = f(x)$ for every $x \in F$.

We have the following theorem relating the filters of F and F/P.

Theorem 3.8. If P and P' are filters of the frame F and $P \subseteq P'$ then

1. P is also a filter of P'
2. $P'/P = \{[x]_P^{P'} \mid x \in P'\}$ is a filter of F/P.

Proof: 1. Follows from the definition of the filter.

2. First we shall show that $P'/P \subseteq F/P$. Let $[x]_P$ and $[x]_P^{P'}$ denote elements of F/P and P'/P respectively containing x. Let $x \in P'$ and $y \in [x]_P$. Then $y \in F$ and $x \sim y$ with respect to P. Hence there exist $a, b \in F$ such that $x \wedge a = y \wedge b$. Since $P \subseteq P'$ and $x \in P'$, $x \wedge a = y \wedge b \in P'$. Since P' is a filter of F, $(y \wedge b) \vee y = y \in P'$. Thus $y \in [x]_P^{P'}$. Hence $[x]_P \subseteq [x]_P^{P'}$. Also clearly $[x]_P^{P'} \subseteq [x]_P$. Therefore $[x]_P = [x]_P^{P'}$. Thus each element of P'/P is also an element in F/P. Now we shall prove that P'/P is a filter of F/P. Let $[x]_P^{P'}$ and $[y]_P^{P'}$ be two elements of P'/P. Then x and y are elements of P' and hence $x \wedge y \in P'$. Thus $[x]_P^{P'} \tilde{\wedge} [y]_P^{P'} = [x]_P \tilde{\wedge} [y]_P = [x \wedge y]_P$ $= [x \wedge y]_P^{P'} \in P'/P$. Also for $[z]_P \in F/P$ and $[x]_P = [x]_P^{P'} \in P'/P$, we have $z \in F$ and $x \in P'$ and hence $z \vee x \in P'$. Thus $[z]_P \tilde{\vee} [x]_P^{P'} = [z]_P \tilde{\vee} [x]_P = [z \vee x]_P = [z \vee x]_P^{P'} \in P'/P$. Hence P'/P is a filter of F/P.

Theorem 3.9. If $\tilde{P}$ is a filter of F/P then $\cup\tilde{P} = \cup\{[x]_P \mid [x]_P \in \tilde{P}\}$ is a filter of F and $P \subseteq \cup\tilde{P}$.

Proof: Let $a, b \in \cup\tilde{P}$. Then there exist $[x]_P$ and $[y]_P \in \tilde{P}$ such that $a \in [x]_P$ and b $\in [y]_P$.

Thus $x \sim a$ and $y \sim b$. Now by lemma 3.2 $x \wedge y \sim a \wedge b$. Hence $a \wedge b \in [x \wedge y]_P$.

Also $[x \wedge y]_P = [x]_P \tilde{\wedge} [y]_P \in \tilde{P}$ as $\tilde{P}$ is a filter of F/P. Thus $a \wedge b \in \cup\tilde{P}$. Now let $z \in F$ and $a \in \cup\tilde{P}$. Then there exist $[x]_P \in \tilde{P}$ such that $a \in [x]_P$ and $[z]_P \in F/P$. Thus $x \sim a$. Now by lemma 3.2 $z \vee x \sim z \vee a$. Hence $z \vee a \in [z \vee a]_P$. Now $[z \vee a]_P = [z]_P \tilde{\vee} [a]_P \in \tilde{P}$ as $\tilde{P}$ is a filter of F/P. Thus $z \vee a \in \cup\tilde{P}$. Hence $\cup\tilde{P}$ is a filter of F.

Also we have $P =$ $[e_F]_P$ from lemma 3.4 and hence $P \in \tilde{P}$ as $\tilde{P}$ is a filter of F/P. Hence $P \subseteq \cup\tilde{P}$.

Example 3.10. Consider the frame F = { X, {a, b}, {a}, {b}, $\varnothing$ } where X= {a, b, c} and the order is set inclusion. P = { X, {a, b}} and P' = { X, {a, b}, {a}} are two filters of F.

We have $[\mathrm{X}]_P$ = { X, {a, b}}= P = $[\{a,b\}]_P$, $[\{a\}]_P = \{\{a\}\}$, $[\{b\}]_P = \{\{b\}\}$, $[\varnothing]_P = \{\varnothing\}$.

Hence $F/P = \{[\mathrm{X}]_P, [\{a\}]_P, [\{b\}]_P, [\varnothing]_P\}$.

Again $P \subseteq P'$ and $[X]_P^{P'}$ = { X, {a, b}} $= [\{a,b\}]_P^{P'} = [\{a,b\}]_P$, $[\{a\}]_P^{P'} = \{\{a\}\} = [\{a\}]_P$.

Hence $P'/P = \{[X]_P^{P'}, [\{a\}]_P^{P'}\} = \{[\mathrm{X}]_P, [\{a\}]_P\}$

Here P'/P is a filter of F/P. Also $\cup P'/P = [\mathrm{X}]_P \cup [\{a\}]_P$ = { X, {a, b}} $\cup$ {{a}}

= { X, {a, b}, {a}}, a filter of F.

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