PRIORITIZATION METHODS AND ERROR ANALYSIS IN ANALYTIC HIERARCHY PROCESS – AN ILLUSTRATION

V.Shyamprasad,
Department of Humanities and Sciences,
Guru Nanak Institutions Technical Campus,
Hyderabad, India,
Email: shyamnow4u@gmail.com

Dr.P.Kousalya,
Department of Humanities and Sciences,
Vignana Bharathi Institute of Technology,
Hyderabad, India,
Email: kousalya29@yahoo.com

***Abstract*:** This paper explains about Analytic Hierarchy Process which deals selection of set of alternatives towards an overall goal. In this generally Prioritization methods are used to find the priority vector weights, three prioritization methods are used to find the priorities of Pair wise Comparison Matrix and different errors are calculated and compared. The three methods are namely (i). Least Square Error Method (LSE) (ii). Logarithmic Least Square Error Method (LLSE) and (iii). Chi-Square Error (CSE) is used to find error to the final Priority Vector and an error analysis was made.

Key words: Analytic Hierarchy Process, Pair wise comparison, Priority Vector.

1. INTRODUCTION

Analytic Hierarchy process (AHP) was introduced by T.L.Saaty. Estimation of priority vector from pair wise comparison matrices is main part in AHP. There are different techniques to obtain priority vector from comparison matrices. Eigen Vector Method was the first method introduced by Saaty, here in this paper three prioritization methods are used to choose a leader for a company with three alternatives and four criteria, three methods are used to find error to the final priority vector and an error analysis was also made.

2. METHODOLOGY

The procedure to carry out the Analytic Hierarchy Process consists the following steps

2.1. Structuring a decision making problem and selection of criteria:

The first step in AHP is to decompose a decision problem into its constituent parts. This structure gives a goal or focus at the topmost level, criteria and sub criteria at the intermediate level. Arranging all the components in a hierarchy gives an overall view of the complex relationships

and helps the decision maker to access whether the elements in each level are of the same magnitude so that they can be compared perfectly.

2.2. Priority setting of the criteria by using pair wise comparison:

Each pair of criteria decision maker has to respond to a question “How much importance is criterion A relative to criterion B?” Rating the priority of the criteria is done by assigning a weight between 1 (Same importance) and 9 (Extreme importance) to the more important criterion, whereas the reciprocal of this value is assigned to the other criterion in the pair. Weights are then normalized and averaged to get an average weight for each criterion.

2.3. Pair Wise comparison of options on each criterion:

For each pair within the criterion the better option is awarded a score, on a scale between 1 (Equally good) and 9 (Extremely better), while the other option in the pairing is given a rating equal to the reciprocal of this value. Each score records how good option “A” reaches criterion “B”. Afterwards ratings are normalized and averaged. Comparisons of elements in pairs require that they are homogeneous or else close with respect to the common attribute; or else significant errors may be introduced into the process of measurement (Saaty 1990).

2.4. Obtaining an overall relative score for each option:

In the final step the option scores are combined with the criterion weights to produce an overall score for each option. Finally, after judgments have been made on the effect of all the elements and priorities have been compared for the hierarchy as a whole, sometimes the less important elements can be eliminated for further consideration because of relatively small impact on the overall objective. Then the priorities can be recomputed either with or without changing the judgments (Saaty 1990).

Hierarchy process is shown in the Fig.1.

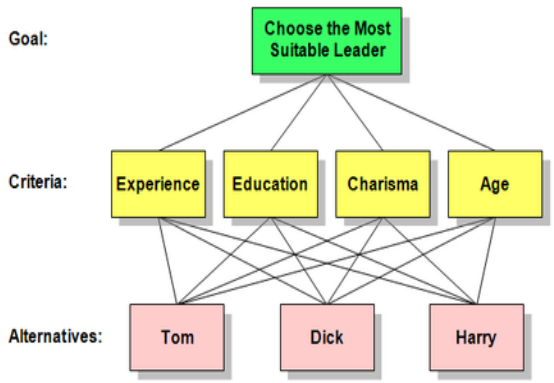

Fig.1 AHP hierarchy

3. PRIORITIZATION METHODS

There are some methods to find the priorities of alternatives, here in this paper three prioritization methods (1). Geometric Mean Method (2). Additive Normalization Method (3). Stochastic Vector Method have been used to find the priorities of the alternatives (Tom, Dick, Horry) for selection of a leader for a company with four criteria (Experience, Education, Charisma and Age) the three errors were also calculated and analyzed.

3.1. Geometric Mean Method (GMM):

This method is used to find the weights to the criteria or alternatives. The pair-wise comparison matrix of alternatives is shown in table1. Where $A_1, A_2 \ldots\ldots\ldots\ldots A_n$ represent the alternatives which are to be ranked. And $a_{11}, a_{12} \ldots\ldots\ldots\ldots\ldots a_{nn}$ show the options of experts. The Geometric Mean Method is explained below which is used to calculate the priority weight vectors.

Table 1: pair-wise comparisons

	A_1	A_2	.....	A_n
A_1	a_{11}	a_{12}	.....	a_{1n}
A_2	a_{21}	a_{22}	.....	a_{2n}
. . . .	. . . .	. . . .	. . . .	. . . .
A_n	a_{n1}	a_{n2}	.....	a_{nn}

Obtain the geometric row means of each row as

$$a_1 = (a_{11} * a_{12} * a_{13} * \ldots\ldots a_{1n})^{\frac{1}{n}}$$
$$a_2 = (a_{21} * a_{22} * a_{23} * \ldots\ldots a_{2n})^{\frac{1}{n}}$$
$$a_n = (a_{n1} * a_{n2} * a_{n3} * \ldots\ldots a_{nn})^{\frac{1}{n}}$$

The normalized vector of $(a_1, a_2 \ldots\ldots\ldots\ldots\ldots a_n)$ becomes the solution vector.

3.2. Additive Normalization Method (ANM):

To obtain the priority vector *w* by this method it is enough to divide the elements of each column of matrix *A* by sum of that column (i.e. normalize the column), then add the elements in each resulting row and finally divide this sum by the number of elements in the row. This method is described by relations (1) and (2).

$$a_{ij}' = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}}, \quad i,j = 1,2,3 \ldots . n \qquad (1)$$

$$w_i = \left(\frac{1}{n}\right)\sum_{i=1}^{n} a_{ij}', i = 1,2,3 \ldots. \qquad (2)$$

3.3. Stochastic Vector Method (SVM) Algorithm:

Step-1: If the PCM is consistent i.e.$a_{ij} = a_{ik}a_{kj}$ for each element, then use GMM and go to Step-6
Step-2: If the PCM is not consistent i.e.$a_{ij} \neq a_{ik}a_{kj}$ for at least one i and j, then divide each row vector by its trace to get a stochastic row vector and let A^s be the stochastic matrix of such rows.
Step-3: Let X_0 be the initial guess stochastic fixed vector and the next vector is obtained by $X_1 = A^s X_0$
Step-4: While the error of $|X_0 - X_1|$ is less than the pre assigned value do $X_1 = A^s X_0$ and $X_0 = X_1$
Step-5: Write "The solution vector by SVM is X_1" Go to Step-7.
Step-6: Write "The solution vector by GMM isX_1"
Step-7: END

4. ILLUSTRATION

Consider an example which explains the use of Analytic Hierarchy Process in selecting a leader whose founder is about to retire. There are several candidates and several criteria for selection of the most suitable one. In order to choose a leader for the company with the help of prioritization methods and to analyze the result, an illustration was taken from AHP literature, Saaty (2008), Chapter 5. Three methods were used to find error to the final priority vector and an error analysis was made.

4.1. Geometric Mean Method (GMM):

Table2

C_1:Experience	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	1/4	4	0.217
Dick	4	1	9	0.717
Harry	¼	1/9	1	0.066

$\lambda_{\max} = 3.0369, C.R = 0.04$

Table2.1

C_2:Education	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	3	1/5	0.188
Dick	1/3	1	1/7	0.081
Harry	5	7	1	0.731

$\lambda_{\max} = 3.0649, C.R = 0.06$

Table2.2

C_3:Charisma	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	5	9	0.743
Dick	1/5	1	4	0.194
Harry	1/9	1/4	1	0.063

$\lambda_{max} = 3.0713, C.R = 0.07$

Table2.3

C_4:Age	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	1/3	5	0.265
Dick	3	1	9	0.672
Harry	1/5	1/9	1	0.063

$\lambda_{max} = 3.0291, C.R = 0.03$

Table2.4: Criteria compared with respect to reaching the goal

Criteria	**C_1**	**C_2**	**C_3**	**C_4**	**P.V**
C_1	1	4	3	7	0.547
C_2	¼	1	1/3	3	0.127
C_3	1/3	3	1	5	0.270
C_4	1/7	1/3	1/5	1	0.057

$\lambda_{max} = 4.1184, C.R = 0.04$

Table2.5: AHP solution by GMM

Criteria / **Alternatives**	**C_1**	**C_2**	**C_3**	**C_4**	**P.V**
Tom	0.119	0.024	0.201	0.015	0.359
Dick	0.392	0.010	0.052	0.038	0.492
Harry	0.036	0.093	0.017	0.004	0.149
Criteria Weights	0.547	0.127	0.270	0.057	1.000

4.2. Additive Normalization Method (ANM):

Table3

C_1:Experience	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	¼	4	0.220
Dick	4	1	9	0.713
Harry	¼	1/9	1	0.067

$\lambda_{max} = 3.0369, C.R = 0.04$

Table3.1

C_2:Education	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	3	1/5	0.193
Dick	1/3	1	1/7	0.083
Harry	5	7	1	0.724

$\lambda_{max} = 3.0649, C.R = 0.06$

Table3.2

C_3:Charisma	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	5	9	0.735
Dick	1/5	1	4	0.199
Harry	1/9	¼	1	0.065

$\lambda_{max} = 3.0713, C.R = 0.07$

Table3.3

C_4:Age	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	1/3	5	0.267
Dick	3	1	9	0.669
Harry	1/5	1/9	1	0.064

$\lambda_{max} = 3.0291, C.R = 0.03$

Table3.4: Criteria compared with respect to reaching the goal

Criteria	**C_1**	**C_2**	**C_3**	**C_4**	**P.V**
C_1	1	4	3	7	0.538
C_2	¼	1	1/3	3	0.132
C_3	1/3	3	1	5	0.271
C_4	1/7	1/3	1/5	1	0.059

Table3.5: AHP solution by ANM

Criteria / **Alternatives**	**C1**	**C2**	**C3**	**C4**	**P.V**
Tom	0.118	0.025	0.199	0.016	0.358
Dick	0.384	0.011	0.054	0.040	0.489
Harry	0.036	0.096	0.018	0.003	0.153
Criteria Weights	0.538	0.132	0.271	0.059	1.000

C. Stochastic Vector Method (SVM)

Table4

C_1:Experience	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	¼	4	0.255
Dick	4	1	9	0.679
Harry	¼	1/9	1	0.066

Table4.1 $\lambda_{max} = 3.0369, C.R = 0.04$

C_2:Education	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	3	1/5	0.225
Dick	1/3	1	1/7	0.079
Harry	5	7	1	0.696

$\lambda_{max} = 3.0649, C.R = 0.06$

Table4.2

C_3:Charisma	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	5	9	0.696
Dick	1/5	1	4	0.241
Harry	1/9	1/4	1	0.063

$\lambda_{max} = 3.0713, C.R = 0.07$

Table4.3

C_4:Age	**Tom**	**Dick**	**Harry**	**Priority Vector**
Tom	1	1/3	5	0.306
Dick	3	1	9	0.628
Harry	1/5	1/9	1	0.066

$\lambda_{max} = 3.0291, C.R = 0.03$

Table4.4: Criteria compared with respect to reaching the goal

Criteria	**C_1**	**C_2**	**C_3**	**C_4**	**P.V**
C_1	1	4	3	7	0.490
C_2	¼	1	1/3	3	0.150
C_3	1/3	3	1	5	0.305
C_4	1/7	1/3	1/5	1	0.055

$\lambda_{max} = 4.1184, C.R = 0.04$

Table4.5: AHP solution by SVM

Criteria Alternatives	**C1**	**C2**	**C3**	**C4**	**P.V**
Tom	0.125	0.034	0.212	0.017	0.388
Dick	0.333	0.012	0.074	0.034	0.453
Harry	0.032	0.104	0.019	0.003	0.158
Criteria Weights	0.490	0.150	0.305	0.055	1.000

5. TYPES OF ERRORS

5.1. Least Square Error

If a_{ij} is comparison between two alternatives $A_i and B_j$ and p_i is the priority if 'i' p_j is the priority of 'j' then least square error is obtained by using $\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{a_{ij} - \left(\frac{p_i}{p_j}\right)\right\}^2$

5.2. Logarithmic Least Square Error

If a_{ij} is comparison between two alternatives $A_i and B_j$ and p_i is the priority if 'i' p_j is the priority of 'j' then logarithmic least square error is obtained by using

$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{log\ a_{ij} - log\ \left(\frac{p_i}{p_j}\right)\right\}^2$

5.3. Chi Square Error

If a_{ij} is an element of a Pair Wise Comparison Matrix and $\widehat{a_{ij}}$ is its estimate then the chi square error is obtained by using $\sum_i \sum_j \frac{(a_{ij} - \hat{a}_{ij})^2}{\hat{a}_{ij}}$

Consider an example of Selection of a leader for a company, where three alternatives Tom, Dick and Harry with four criteria Experience, Education, Charisma and Age. In this paper three prioritization methods used to find the final priority vector and three methods were used to find error to each criteria of the final priority vector, the errors were as follows:

Errors by Geometric Mean Method (GMM)

Table5

Criteria	**LSE**	**LLSE**	**CSE**
Experience	0.603	1.452	0.379
Education	22.975	12.199	0.079
Charisma	4.667	5.343	0.199
Age	120.141	20.289	0.026

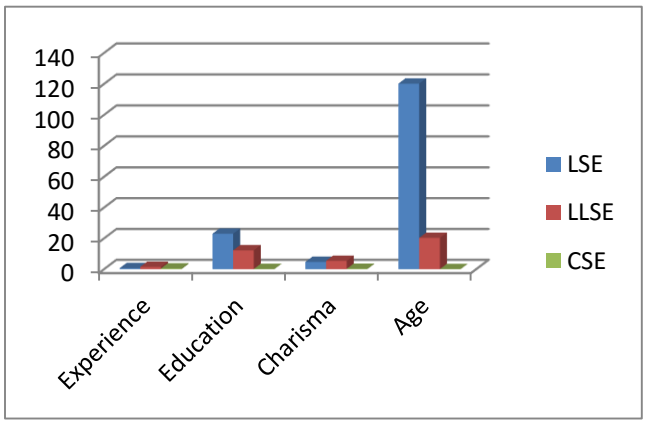

Fig .2

Errors by Additive Normalization Method (ANM)

Table6

[Criteria	**LSE**	**LLSE**	**CSE**
Experience	0.638	1.674	0.364
Education	21.993	11.7	0.074
Charisma	4.622	5.237	0.427
Age	111.365	20.641	0.030

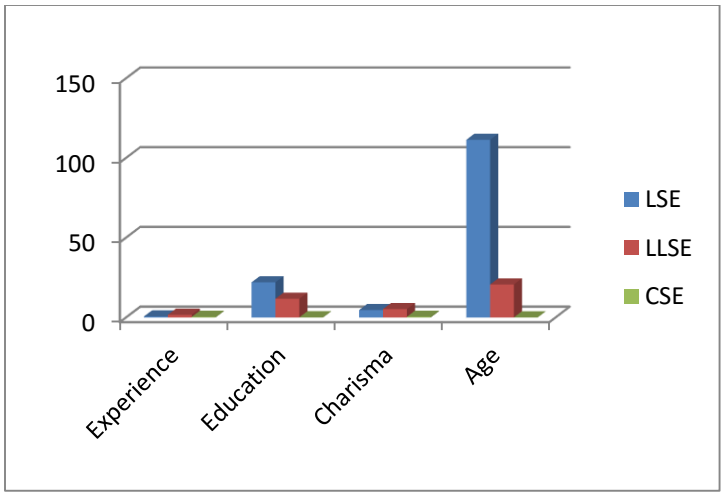

Fig.3

Errors by Stochastic Vector Method (SVM)

Table7

Criteria	**LSE**	**LLSE**	**CSE**
Experience	0.879	1.847	0.752
Education	16.465	10.314	0.227
Charisma	3.365	4.364	0.183
Age	125.041	21.426	0.062

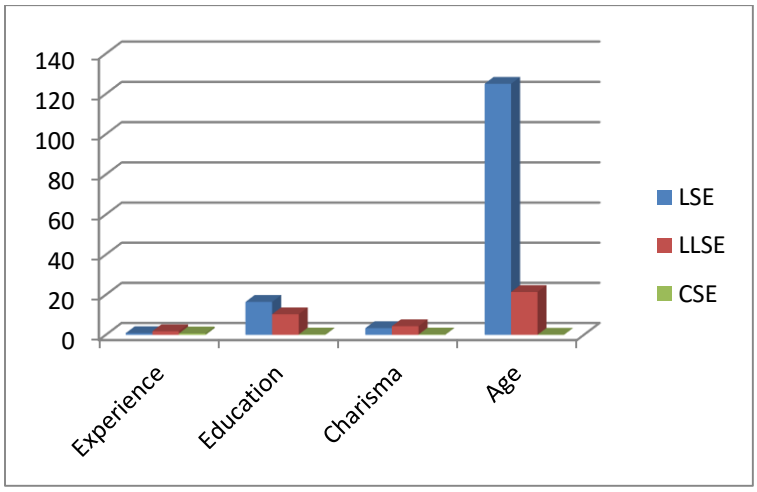

Fig.4

6. CONCLUSION

It can be observed that alternative Dick is ranked first followed by Tom and Harry and the rankings of the alternatives by three methods are same though the weights of the elements are differing in all the three prioritization methods. Here an error analysis is made and it was observed that LSE for the criteria four (Age) was highest and that was lowest for criteria one (Experience) , LLSE for the criteria four (Age) was highest and that was lowest for the criteria one (Experience), in all the methods , CSE for the criteria one (Experience) was highest in GMM, criteria three (Charisma) was highest in ANM, criteria one (Experience) was highest in SVM and that was lowest for criteria four (Age) in all the three methods.

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