AVERAGE NUMBER OF REAL ZEROS OF RANDOM TRIGONOMETRIC POLYNOMIAL

DR. P. K. MISHRA & DIPTY RANI DHAL

Associate professor of Mathematics,
CET, BPUT, BBSR, ODISHA, INDIA

mishrapkdr@gmail.com & dipty24dhal@gmail.com

Abstract:- Let EN(T; Φ' , Φ'') denote the average number of real zeros of the random trigonometric polynomial

$$T=T_n(\Phi, \omega)=\sum_{K=1}^{n} a_K(\omega) b_K \cos k\theta$$

In the interval (Φ' , Φ''). Assuming that $a_k(\omega)$ are independent random variables identically distributed according to the normal law and that $b_k = k^p$ ($p \geq 0$) are positive constants, we show that

$$EN(T : 0, 2\pi) \sim \left\{\left(\frac{2p+1}{2p+3}\right)(1-\epsilon_n{}^2)\right\}^{1/2} 2n + O(\log n)$$

Outside an exceptional set of measure at most (2/ n) where

$$\epsilon_n{}^2 = \frac{4\beta^2(2p+1)(2p+3)}{SS'(\log n)^2}$$

β = constant $\qquad S \sim 1 \qquad S' \sim 1$

1991 Mathematics subject classification (amer. Math. Soc.): 60 B 99.

Keywords and phrases: Independent, identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots, domain of attraction of the normal law, slowly varying function.

1 . Introduction

Let N(T ; Φ' , Φ'') be the number of real zeros of trigonometric polynomial T = T_n (Φ, ω) = $\sum_{K=1}^{n} a_K(\omega) b_K \cos k\theta$ **(1)**

In the interval (Φ' , Φ'') where the coefficients $a_k(\omega)$ are mutually independent random variables identically distributed according to the normal law ; $b_k=k^p$ are positive constants and when multiple zeros are counted only once . Let EN (T ; Φ' , Φ'') denote the expectation of N (T ; Φ' , Φ''). Obviously , T_n (Φ, ω) can have at most 2n most zeros in the interval (0 , 2π) Das [1] studied the class of polynomials

$$\sum_{K=1}^{n} k^p \left(g_{2K-1}\cos k\theta + g_{2k}\sin k\theta\right) \qquad \textbf{(2)}$$

where g_k are independent normal random variables for fixed p > -1/2 and proved that in the interval (0 , 2π) the function (2) have

$$\left[(2p+1)/(2p+3)\right]^{1/2} 2n + O(n^{11/13+4q/13+\in 1})$$

number of real roots when n is large . Here , q= max(0, -p) and $\in_1 < (2/13)(1-2q)$.. The measure of exceptional set does not exceed $n^{-2\in 1}$

Das [2] took the polynomial (1) where $a_K(\omega)$ are independent normal random variables identically distributed with mean zero and variance one . He proves that in the interval 0 ≤ θ ≤ 2π , the average number of real zeros of polynomials (1) is

$$[(2p+1) / (2p+3)]^{1/2} 2n+O(n) \qquad \textbf{(3)}$$

for $b_K = k^p$(p > -1/2) and of the order of $n^{p+3/2}$ if -3/2 ≤ p ≤ -1/2 for large n.

In this paper we consider the polynomial (1) with conditions as in **DAS [2]** and use the Kac_Rice formula for the expectation of the number of real zeros and obtain that for p ≥ 0

$$\text{EN (T ; 0 , 2 π)} \sim \left\{\frac{2p+1}{2p+3}\left(1-\in_n{}^2\right)\right\}^{1/2} 2n + O(\log n)$$

Where

$$\in_n^{\ 2} = \frac{4\beta^2(2p+1)(2p+3)}{SS'(\log n)^2} \qquad \beta = \text{constant} \qquad S \sim 1 \qquad S' \sim 1$$

Our asymptotic estimate implies that Das's estimate in [1] is approached from below . Also our term is smaller .

The particular case for p=0 has been considered by **Dunnage [3]** and **Pratihari and Bhanja [4]** . Dunnage has shown that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions $T_n(\Phi, \omega)$ have

$$\frac{2n}{\sqrt{3}} + O\left(n^{11/13}(\log n)^{3/13}\right) \qquad \textbf{(4)}$$

zeros when n is large . The measure of the exceptional set does not exceed $(\log n)^{-1}$. Using the kac_rice formula we tried to obtain in [4] that

$$EN(T;\ 0, 2\pi) \sim \frac{2n}{\sqrt{6}} + O(\log n)$$

Professor Dunnage[5] comments that our result is incorrect .he is quite right when he says than an asymptotic estimate is unique and that both results (4) and (5) cannot be correct . But in his calculations given in paragraph 4 of [5] he seems to have imported a factor 2 and the correct calculation would give $I' \sim 2\pi n / \sqrt{3}$. accepting his own statement in paragraph 3 that $I < I'$, our point is clear . However , since $I' \sim 2\pi n / \sqrt{3}$ on direct integration , our estimation of EN as found in [4] , contained in the statement (5) above , must be wrong . We are sorry about our mistake . In this paper we consider our original integral I and evaluate it directly instead of placing it between two integrals as in [4] , the second one being possibly suspect . This rectification eventually raises our estimate for EN but , all the same , keeps it below Dunnage's estimate stated in above . The purport of our result is that EN approaches the value $2n / \sqrt{3}$ from below . This is something meaningful . We prove the following theorem .

Theorem . The average number of real zeros in the interval $(0, 2\pi)$ of the class of random trigonometric polynomials of the form

$$\sum_{K=1}^{n} a_K(\omega) b_K \cos k\theta$$

where $a_K(\omega)$ are mutually independent random variables identically distributed according to normal law with mean zero and variance one and $b_K = k^p$ $(p \geq 0)$ are positive constants, is asymptotically equal to

$$\left\{ \frac{2p+1}{2p+3} \left(1 - \in_n{}^2\right) \right\}^{1/2} 2n + O(\log n) \qquad \textbf{(5)}$$

outside an exceptional set of measure at most (2 / n) where

$$\in_n{}^2 = \frac{4\beta^2(2p+1)(2p+3)}{SS'(\log n)^2} \qquad \beta = \text{constant} \qquad S \sim 1 \qquad S' \sim 1$$

2 . The Approximation for EN (T ; 0 , 2π)

Let L (n) be a positive-valued function of n such that L(n) and n/ L(n) both approach infinity with n . We take $\in = L(n)/n$ throughout.

Outside a small exceptional set of ω, $T_n(\theta, \omega)$ has a negligible number of zeros in each of the intervals $(0, \in)$, $(\pi - \in, \pi + \in)$ and $(2\pi - \in, 2\pi)$. By periodicity , of zeros in each of intervals $(0, \in)$ and $(2\pi - \in, 2\pi)$ is the same as number in $(-\in, \in)$. We shall use the following lemma , which is due to Das [2] .

Lemma. The probability that T has more than $1 + (\log n)^{-1}(\log n + \log D_n + 4n\in)$ Zeros in $\omega - \in \leq \theta \leq \omega + \in$ does not exceed $2\exp(-n\in)$,where

$$D_n = \sum_{K=1}^{n} b_K$$

The steps in this section follow closely those in section 2 of [4] . therefore , we indicate only the modifications necessary . In this case we have

T= $\sum_{K=1}^{n} a_K(\omega) b_K \cos k\theta$ T'= $-\sum_{K=1}^{n} k a_K(\omega) b_K \sin k\theta$

$$\phi(y,z) = \exp\left\{-\frac{1}{2}\sum_{K=1}^{n}(y\cos\theta - zk\sin k\theta)^2 b_K{}^2\right\}$$

$$p(0,\eta) = \frac{1}{(2\Pi)^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \exp(i\eta z)\exp\left\{-\frac{1}{2}\sum_{K=1}^{n}(y\cos k\theta - zk\sin k\theta)^2 b_K{}^2\right\} dy$$

And finally

$$\int_{-\infty}^{\infty}|\eta| p(0,\eta) d\eta = \frac{1}{2\Pi^2}\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{K=1}^{n}(u\cos k\theta + k\sin k\theta)^2 b_K{}^2}{(Au+B)^2}\right\} du \qquad \textbf{(6)}$$

for fixed non-zero real constants A and B to be chosen .

3 . Estimation of the integral of equation (6)

Consider the integral

$$I = \int_{-\infty}^{\infty}\log\left\{\frac{\sum_{K=1}^{n}(u\cos k\theta + k\sin k\theta)^2 b_K{}^2}{(Au+B)^2}\right\} du$$

Which exists in general as a principal value if

$$A^2 = \sum_{K=1}^{n} b_K^2 \cos^2 k\theta$$

Let
$$B^2 = \sum_{K=1}^{n} k^2 b_K^2 \sin^2 k\theta$$

And
$$C^2 = \sum_{K=1}^{n} k b_K^2 \cos k\theta \sin k\theta$$

As in Das[2] ,letting $b_k = k^p (p \geq 0)$ we get

$$A^2 = \frac{1}{2}\frac{n^{2P+1}}{2p+1}\left\{1 + O\left(\frac{1}{L(n)}\right)\right\} = \frac{1}{2} S \frac{n^{2p+1}}{2p+1}$$

$$B^2 = \frac{1}{2}\frac{n^{2P+3}}{2p+3}\left\{1 + O\left(\frac{1}{L(n)}\right)\right\} = \frac{1}{2} S' \frac{n^{2p+3}}{2p+3}$$

$$C^2 = O\left(\frac{n^{2p+2}}{L(n)}\right) = \frac{\beta n^{2p+2}}{L(n)}$$

Say (β = constant)

outside the set { 0 , ± π , ± 2π, … } of the values of θ , AB > C^2 . We have

$$I = \int_{-\infty}^{\infty} \log\left\{\frac{\sum_{K=1}^{n}(u\cos k\theta + k\sin k\theta)^2 b_K{}^2}{(Au+B)^2}\right\} du$$

$$= \int_{-\infty}^{\infty} \log\left\{\frac{u^2 + 2c^2u + b^2}{u^2 + 2bu + b^2}\right\} du$$

Where c=(C/A) and b=(B/A) . now by integration by parts ,

$$\int_0^X \log(u^2 + 2c^2u + b^2)du = \left[u\log(u^2 + 2c^2u + b^2)\right]_0^X - \int_0^X \frac{2u^2 + 2c^2u}{u^2 + 2c^2u + b^2} du$$

$$= 2X\log X + 2c^2 - 2X + 2\int_0^X \frac{-c^2u + b^2}{u^2 + 2c^2u + b^2} du + O\left(\frac{1}{X}\right)$$

and

$$\int_{-X}^{0} \log(u^2 + 2c^2u + b^2)du = \int_0^X \log(u^2 - 2c^2u + b^2)du$$

$$= 2X\log X + 2c^2 - 2X + 2\int_0^X \frac{-c^2u + b^2}{u^2 - 2c^2u + b^2} du + O\left(\frac{1}{X}\right)$$

Therefore

$$\int_{-X}^{X} \log(u^2 + 2c^2u + b^2)du = 4X\log X - 4X + 2\int_{-X}^{X} \frac{c^2u + b^2}{u^2 + 2c^2u + b^2} du + O\left(\frac{1}{X}\right)$$

Again

$$\int_{-X}^{X} \log\left(u^2 + 2bu + b^2\right)du = 4X\log X - 4X + 2\int_{-X}^{X} \frac{bu + b^2}{u^2 + 2bu + b^2}du + O\left(\frac{1}{X}\right)$$

Hence the integral

$$\int_{-X}^{X} \log\left\{\frac{u^2 + 2c^2u + b^2}{u^2 + 2bu + b^2}\right\}du = 2\int_{-X}^{X} \frac{c^2u + b^2}{u^2 + 2c^2u + b^2}du - 2\int_{-X}^{X} \frac{bu + b^2}{u^2 + 2bu + b^2} + O\left(\frac{1}{X}\right)$$

$$2\int_{-X}^{X} \frac{c^2u + b^2}{u^2 + 2c^2u + b^2}du = c^2\left[\log\left(u^2 + 2c^2u + b^2\right)\right]_{-X}^{X} + 2\sqrt{b^2 - c^4}\left[\tan^{-1}\frac{u + c^2}{\sqrt{b^2 - c^4}}\right]_{-X}^{X}$$

$$= c^2\log\left(\frac{X^2 + 2c^2X + b^2}{X^2 - 2c^2X + b^2}\right) + 2\sqrt{b^2 - c^4}\tan^{-1}\left(\frac{2X\sqrt{b^2 - c^4}}{b^2 - X^2}\right)$$

When $X \to \infty$, we have

$$\int_{-\infty}^{\infty} \frac{c^2u + b^2}{u^2 + 2c^2u + b^2}du = \Pi\sqrt{b^2 - c^4}$$

$$\int_{-\infty}^{\infty} \frac{bu + b^2}{u^2 + 2bu + b^2}du = 0$$

and $O\left(\frac{1}{X}\right) = 0$

Therefore

$$I = \lim_{X \to \infty}\int_{X}^{X} \ldots = \int_{-\infty}^{\infty} \log\left\{\frac{u^2 + 2c^2u + b^2}{u^2 + 2bu + b^2}\right\}du$$

$$= 2\Pi\left(b^2 - c^4\right)^{1/2} = 2\Pi\left(\frac{A^2B^2 - C^4}{A^4}\right)^{1/2}$$

$$= \left\{\left(\frac{S'}{S}\right)\left(\frac{2p+1}{2p+3}\right)\left(1+ \in_n{}^2\right)\right\}^{1/2} 2\Pi n$$

Where

$$\in_n{}^2 = \frac{4\beta^2(2p+1)(2p+3)}{SS'(L(n))^2} \qquad S \sim 1 \qquad S' \sim 1$$

Thus

$$I \sim \left[(2p+1)/(2p+3)\right]^{1/2} 2\Pi n\left(1- \in_n{}^2\right)^{1/2}$$

Now the result follows from the section 4 of [4] , choosing L(n) =log n. the cases where $-1/2 < p < 0$ and $p = -1/2$ can be similarly dealt with and results can be obtained to show that Das's estimates are approached from below .

We express our reverence and deep gratitude to Sir A.C.offord for his approval of the thesis [6] which presents this result , and for his valuable suggestions .

REFERENCES

1. **Das, M.K.** Real zeros of a random sum of orthogonal polynomials, ***Proc. Amer. Math. Soc. 27, 1 (1971), 147-153.***

2. **Das, M.K.** Real zeros of a class of random algebraic Polynomials, ***Journal of Indian Math. Soc.. 36 (1972), 53-63.***

3. **Dunnage, J.E.A.** The number of real zeros of a class of random algebraic polynomials (I), ***Proc. London Math. Soc. (3) 18 (1968), 439-460.***

4. **Pratihari and Bhanja .**On the number of real zeros of random trigonometric polynomial, ***Trans. Of Amer. Math. Soc. 238 (1978) 57-70.***