Optimal Convex Combination Bounds of Arithmetic and Second Seiffert Means for Neuman-Sándor Mean

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Abstract. In this paper, we present the least value $\alpha$ and the greatest value $\beta$ such that the double inequality

$$\alpha A(a, b) + (1 - \alpha)T(a, b) < M(a, b) < \beta A(a, b) + (1 - \beta)T(a, b)$$

hold for all $a, b > 0$ with $a \neq b$, where $A(a, b)$, $M(a, b)$ and $T(a, b)$ are the arithmetic, Neuman-Sándor and second Seiffert means of $a$ and $b$, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a, b)[1]$ was defined by

$$M(a, b) = \frac{a - b}{2 \sinh^{-1} \left( \frac{a - b}{a + b} \right)},$$

where $\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a, b)$ can be found in the literature [1, 2].

Let $H(a, b) = (2ab)/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (a - b)/(\log a - \log b)$, $P(a, b) = (a - b)/(4 \tan^{-1} \sqrt{2} - \pi)$, $A(a, b) = (a + b)/2$, $T(a, b) = (a - b)/(2 \tan^{-1} (a - b)/(a + b))$, $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ and $C(a, b) = (a^2 + b^2)/(a + b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic means of $a$ and $b$, respectively. Then

$$\min \{a, b\} < H(a, b) < G(a, b) < L(a, b) < P(a, b) < A(a, b)$$

$$< M(a, b) < T(a, b) < Q(a, b) < C(a, b) < \max \{a, b\}$$

hold for all $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 2] proved that the inequalities

$$\frac{\pi}{4 \log (1 + \sqrt{2})} I(a, b) < M(a, b) < \frac{A(a, b)}{\log (1 + \sqrt{2})},$$

$$\sqrt{2T^2(a, b) - Q^2(a, b)} < M(a, b) < \frac{T^2(a, b)}{Q^2(a, b)},$$

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### 2. Lemmas

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**Lemmas 1.** Let \( \mu = 1/(4 - \pi)[4 - \pi/\log(1 + \sqrt{2})] = 0.5074 \ldots, p \in \{1/2, \mu\}, \) and \( \omega_p(t) = (p - 1)^2t^4 + (1-p)^2(1-10p)t^3 + (1-p)(8p^2 - 14p + 1)t^2 + (4p^2 - 2p + 3)t + 2(2p + 1). \) Then \( \omega_p(t) > 0 \) holds for all \( t \in (0, 1) \).

**Proof.** Simple computations lead to

\[
\lim_{t \to 0^+} \omega_p(t) = 2(2p + 1) > 0, \quad \lim_{t \to 1^-} \omega_p(t) = (2 - p)(19p^2 - 12p + 4) > 0, \quad (2.1)
\]

\[
\lim_{t \to 0^+} \omega_p'(t) = 4p^2 - 2p + 3 > 0, \quad \lim_{t \to 1^-} \omega_p'(t) = (3 - 2p)(25p^2 - 24p + 4) < 0, \quad (2.2)
\]

\[
\lim_{t \to 0^+} \omega_p''(t) = 2(1-p)(8p^2 - 14p + 1) < 0, \quad (2.3)
\]

and

\[
\omega_p'''(t) = 6[4(p-1)^2t^3 + (1-p)^2(1-10p)] < 0
\]

for \( t \in (0, 1). \) (2.3) and (2.4) imply that \( \omega_p'(t) \) is strictly decreasing in \( (0, 1) \). It follows from (2.2) and the monotonicity of \( \omega_p(t) \) that there exists \( t_0 \in (0, 1) \) such that \( \omega_p(t_0) > 0 \) for \( t \in (0, t_0) \) and \( \omega_p(t_0) < 0 \) for \( t \in (t_0, 1) \), hence \( \omega_p(t) \) is strictly increasing in \( (0, t_0) \) and strictly decreasing in \( (t_0, 1) \). Therefore the conclusion of lemma 1 is educed from (2.1) the monotonicity of \( \omega_p(t) \). \( \square \)

**Lemmas 2.** Let \( \mu = 1/(4 - \pi)[4 - \pi/\log(1 + \sqrt{2})] = 0.5074 \ldots, p \in \{1/2, \mu\}, \) and \( v_p(t) = 2[2(1-p)^2t^3 + 5(1-p)^2t^2 + 2(p^2 - 3p + 1)t - (2p + 1)]. \) Then \( v_p(t) < 0 \) holds for all \( t \in (0, 1). \)

**Proof.** Simple computations yield

\[
\lim_{t \to 0^+} v_p(t) = -2(2p + 1) < 0, \quad \lim_{t \to 1^-} v_p(t) = 2(2-p)(9p - 4) < 0, \quad (2.5)
\]

\[
\lim_{t \to 0^+} v_p'(t) = 4(p^2 - 3p + 1) < 0, \quad \lim_{t \to 1^-} v_p'(t) = 4(9p^2 - 19p + 9) > 0, \quad (2.6)
\]

and

\[
v_p''(t) = 4(1-p)^2(6t + 5) > 0
\]

holds for all \( t \in (0, 1). \) From (2.7) we know that \( v_p'(t) \) is strictly increasing in \( (0, 1) \).

It follows from (2.6) and the monotonicity of \( v_p'(t) \) that there exists \( t_1 \in (0, 1) \) such that \( v_p'(t) < 0 \) for \( t \in (0, t_1) \) and \( v_p'(t) > 0 \) for \( t \in (t_1, 1) \), hence \( v_p(t) \) is strictly decreasing in \( (0, t_1) \) and strictly increasing in \( (t_1, 1) \). Therefore the conclusion of lemma 2 is elicited from (2.5) and the monotonicity of \( v_p(t) \). \( \square \)

**Lemmas 3.** Let \( \mu = 1/(4 - \pi)[4 - \pi/\log(1 + \sqrt{2})] = 0.5074 \ldots, \) and \( L_\mu(t) = (1 - \mu)^6t^7 + 2(1-\mu)^4(10\mu^2 - 11\mu - 7)t^6 + (1-\mu)^3(116\mu^2 - 48\mu + 93)t^5 + 4(1-\mu)^2(40\mu^4 - 116\mu^3 + 36\mu^2 + 99\mu - 51)t^4 + (1-\mu)^2(64\mu^4 - 304\mu^3 + 40\mu^2 + 480\mu - 185)t^3 - 2(32\mu^3 - 16\mu^2 - 240\mu^3 + 398\mu^2 - 181\mu + 15)t^2 + (64\mu^4 - 336\mu^3 + 380\mu^2 - 16\mu - 53)t + 8(1 + 2\mu)(1 - 2\mu)(3 - 2\mu). \) Then there exists \( \eta_2 \in (0, 1) \) such that \( L_\mu(t) < 0 \) for \( t \in (0, \eta_2) \) and \( L_\mu(t) > 0 \) for \( t \in (\eta_2, 1). \)

**Proof.** By calculating first-sixth derived functions of \( L_\mu(t) \) and the numerical computations we know that \( L_\mu^{(6)}(t) < 0 \) for \( t \in (0, 1) \), and \( L_\mu(0) < 0 \), \( L_\mu(1) > 0 \), \( L_\mu(0) > 0 \), \( L_\mu(1) > 0 \), \( L_\mu^{(4)}(0) > 0 \), \( L_\mu^{(1)}(0) < 0 \), \( L_\mu''(0) > 0 \), \( L_\mu''(1) < 0 \), \( L_\mu''''(0) > 0 \), \( L_\mu''''(1) < 0 \), \( L_\mu'''(0) < 0 \), \( L_\mu'''(1) > 0 \), \( L_\mu^{(4)}(0) < 0 \), \( L_\mu^{(5)}(0) < 0 \), \( L_\mu^{(5)}(1) > 0 \). Apparently \( L_\mu^{(4)}(0) < 0 \), \( L_\mu^{(3)}(0) < 0 \) and \( L_\mu^{(6)}(t) < 0 \) imply that \( L_\mu''(t) \) is strictly decreasing in \( (0, 1) \).

It follows from \( L_\mu''(0) > 0 \) and \( L_\mu''(1) < 0 \) together with the monotonicity of \( L_\mu''(t) \) that there exists \( \eta_0 \in (0, 1) \) such that \( L_\mu''(t) > 0 \) for \( t \in (0, \eta_0) \) and \( L_\mu''(t) < 0 \) for \( t \in (\eta_0, 1) \), so
$L''_\mu(t)$ is strictly increasing in $(0, \eta_0)$ and strictly decreasing in $(\eta_0, 1)$. From $L''_\mu(0) > 0$ and $L''_\mu(1) < 0$ together with the monotonicity of $L''_\mu(t)$ we know that there exists $\eta_1 \in (\eta_0, 1)$ such that $L''_\mu(t) > 0$ for $t \in (0, \eta_1)$ and $L''_\mu(t) < 0$ for $t \in (\eta_1, 1)$, hence $L'_\mu(t)$ is strictly increasing in $(0, \eta_1)$ and strictly decreasing in $(\eta_1, 1)$. $L'_\mu(0) > 0$ and $L'_\mu(1) > 0$ together with the monotonicity of $L'_\mu(t)$ imply that $L'_\mu(t) > 0$ for $t \in (0, 1)$, thus $L_\mu(t)$ is strictly increasing in $(0, 1)$. Therefore the conclusion of lemma 3 follows from $L'_\mu(0) < 0$ and $L'_\mu(1) > 0$ together with the monotonicity of $L'_\mu(t)$.

\[ \square \]

3. Main Results

**Theorem.** The double inequality

$$\alpha A(a, b) + (1 - \alpha)T(a, b) < M(a, b) < \beta A(a, b) + (1 - \beta)T(a, b)$$

(3.1)

holds true for $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 1/(4 - \pi)[4 - \pi/\log(1 + \sqrt{2})] = 0.5074\ldots$ and $\beta \leq 1/2$.

**Proof.** Let $\mu = 1/(4 - \pi)[4 - \pi/\log(1 + \sqrt{2})] = 0.5074\ldots$. Firstly we prove that

\[ \frac{1}{2}[A(a, b) + T(a, b)] > M(a, b), \]

(3.2)

and

\[ \mu A(a, b) + (1 - \mu)T(a, b) < M(a, b). \]

(3.3)

Without loss of generality, we assume that $a > b > 0$. Let $x = (a - b)/(a + b) \in (0, 1)$ and $p \in \{1/2, \mu\}$. Then

\[ \frac{M(a, b)}{A(a, b)} = \frac{x}{\sinh^{-1}(x)}, \quad \frac{T(a, b)}{A(a, b)} = \frac{x}{\tan^{-1}x}, \]

(3.4)

and

\[ \frac{pA(a, b) + (1 - p)T(a, b) - M(a, b)}{A(a, b)} = \frac{E_p(x)}{\log(x + \sqrt{1 + x^2}) \tan^{-1}x}. \]

(3.5)

where

\[ E_p(x) = p \tan^{-1}x \log(x + \sqrt{1 + x^2}) + (1 - p) \log(x + \sqrt{1 + x^2}) - x \tan^{-1}x. \]

(3.6)

Some tedious, but not difficult, calculations lead to

\[ \lim_{x \to 0^+} E_p(x) = 0, \]

(3.7)

\[ \lim_{x \to 1^{-}} E_p(x) = [(\pi/4 - 1)p + 1]\log(1 + \sqrt{2}) - \pi/4, \]

(3.8)

\[ E_p'(x) = \frac{[1 + (1 - p)x^2]G_p(x)}{1 + x^2}, \]

(3.9)

where

\[ G_p(x) = \frac{p(\tan^{-1}x - px + x)\sqrt{1 + x^2} - (1 + x^2)\tan^{-1}x - x}{1 + (1 - p)x^2} + \log(x + \sqrt{1 + x^2}), \]

(3.10)

\[ \lim_{x \to 0^+} G_p(x) = 0, \]

(3.11)

\[ \lim_{x \to 1^-} G_p(x) = \log(1 + \sqrt{2}) + \frac{(\pi - 4)\sqrt{2}p + 2(\sqrt{2} - \pi - 2)}{4(2 - p)}, \]

(3.12)

\[ G_p'(x) = \frac{px[(1 - 2p) + (1 - p)x^2 + 2\sqrt{1 + x^2}]H_p(x)}{[1 + (1 - p)x^2]v^2\sqrt{1 + x^2}}, \]

(3.13)
where

$$H_p(x) = \frac{(1-p)x^4 + (3-2p^2-p)x^2 - 2\sqrt{1+x^2} + 2}{px[(1-2p) + (1-p)x^2 + 2\sqrt{1+x^2}]} - \tan^{-1} x,$$

$$\lim_{x \to 0^+} H_p(x) = 0,$$  

$$\lim_{x \to 1^-} H_p(x) = \frac{2(3-\sqrt{2}) - p(p+3)}{p(2+2\sqrt{2}-3p)} - \frac{\pi}{4},$$

and

$$H_p'(x) = \frac{K_p(x)}{px^2(1+x^2)[(1-2p) + (1-p)x^2 + 2\sqrt{1+x^2}]},$$

where

$$K_p(x) = (p-1)x^3 + (1-p)^2(1-10p)x^6 + (1-p)(8p^2 - 14p + 1)x^4$$

$$+ (4p^2 - 2p + 3)x^2 + (2p + 1) + [4(p-1)^2x^2 + 10(1-p)^2]$$

$$K_p(x) = (1-10p)x^4 + 4(p^2 - 3p + 1)x^2 - 2(2p + 1)\sqrt{1+x^2}.$$

Let $x = \sqrt{t}$ ($t \in (0, 1)$), then

$$K_p(x) = \omega_p(t) + v_p(t)\sqrt{1+t} = \frac{tL_p(t)}{\omega_p(t) - v_p(t)\sqrt{1+t}},$$

where $\omega_p(t) and v_p(t)$ are defined as in lemmas 1 and 2, respectively, and

$$L_p(t) = (1-p^6t^7 + 2(1-p^4)(10p^2 - 11p - 7)t^6 + (1-p)^4(116p^2 - 48p - 93)t^5$$

$$+ 4(1-p)^2(40p^4 - 116p^2 + 36p^3 + 99p - 51)t^4 + (1-p)^2(64p^4 - 304p^2 + 40p^2 + 480p^2 - 185)t^3$$

$$- (1-10p)x^4 + 4(p^2 - 3p + 1)x^2 - 2(2p + 1)\sqrt{1+x^2}.$$ (3.14)

Now we distinguish between two cases:

Case 1. $p = 1/2$. (3.20) leads to

$$L_{1/2}(t) = \frac{1}{64}(t+2)^2[t^4 + 84t^2(1-t) + 104t(1-t) + 8(3t + 8)] > 0,$$ (3.21)

holds for all $t \in (0, 1)$. This fact and (3.19), (3.17) together with lemmas 1 and 2 imply that $H_{1/2}(x) > 0$ for $x \in (0, 1)$, hence $H_{1/2}(x)$ is strictly increasing in $(0, 1)$. Therefore the inequality (3.2) follows from (3.5), (3.7), (3.9), (3.11), (3.13) and (3.15) together with the monotonicity of $H_{1/2}(x)$.

Case 2. $p = \mu$. Here (3.20) becomes $L_p(t)$, which is defined as in lemma 3. By (3.19) and the conclusions of lemmas 1 - 3 we confirm that $K_p(x) < 0$ for $x \in (0, x_0)$ and $K_p(x) > 0$ for $x \in (x_0, 1)$, where $x_0 = \sqrt{\mu}$. This fact and (3.18) imply that $H_p'(x) < 0$ for $x \in (0, x_0)$ and $H_p(x) > 0$ for $x \in (x_0, 1)$, hence $H_p(x)$ is strictly decreasing in $(0, x_0)$ and strictly increasing in $(x_0, 1)$.

Notice that (3.8), (3.12) and (3.16) become

$$\lim_{x \to 1^-} E_p(x) = 0, \quad \lim_{x \to 1^-} G_p(x) = 0.0033 \cdots > 0,$$

$$\lim_{x \to 1^-} H_p(x) = 0.0442 \cdots > 0,$$ (3.22)

respectively. It follows from (3.22), (3.15), (3.13), (3.11), (3.9) and (3.7) together with the monotonicity of $H_p(x)$ that

$$E_p(x) < 0 \quad \text{for} \quad x \in (0, 1).$$ (3.23)

Then the inequality (3.3) follows from (3.5) and (3.23).

Finally, we prove that $\mu A(a, b) + (1-\mu)T(a, b)$ is the best possible lower convex combination bound and $1/2[A(a, b) + T(a, b)]$ is the best possible upper convex combination bound of the...
arithmetic and the second Seiffert means for the Neuman-Sándor mean.

Equations (3.4) lead to
\[
\frac{T(a, b) - M(a, b)}{T(a, b) - A(a, b)} = \frac{x / \tan^{-1} x - x / \sinh^{-1}(x)}{x / \tan^{-1} x - 1} = R(x). \quad (3.24)
\]
From (3.23) one has
\[
\lim_{x \to 1^-} R(x) = \mu, \quad (3.25)
\]
and
\[
\lim_{x \to 0^+} R(x) = \frac{1}{2}. \quad (3.26)
\]

If \( \alpha < \mu \), then (3.24) and (3.25) lead to the conclusion that there exists \( 0 < \delta_1 < 1 \) such that \( M(a, b) < \alpha A(a, b) + (1 - \alpha) T(a, b) \) for all \( a, b > 0 \) with \( (a - b)/(a + b) \in (\delta_1, 1) \).

If \( \beta > 1/2 \), then (3.24) and (3.26) lead to the conclusion that there exists \( 0 < \delta_2 < 1 \) such that \( M(a, b) > \beta A(a, b) + (1 - \beta) T(a, b) \) for all \( a, b > 0 \) with \( (a - b)/(a + b) \in (0, \delta_2) \).

\( \square \)

**Remark.** In [7], we proved that the double inequality
\[
\frac{\alpha G(a, b) + (1 - \alpha) T(a, b)}{M(a, b)} < \frac{\beta G(a, b) + (1 - \beta) T(a, b)}{M(a, b)} \quad (3.27)
\]
holds true for \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha \geq 1/5 \) and \( \beta \leq 1 - \pi/[4 \log(1 + \sqrt{2})] = 0.108893 \cdots \).

The bounds in the double inequalities (3.1) and (3.27) are not comparable to each other. In fact, if we let \( a > b > 0 \) and \( x = \sqrt{a/b} > 1 \), and notate \( \lambda = 1 - \pi/[4 \log(1 + \sqrt{2})] \) and \( \omega = 1/(4 - \pi)\)\( [4 \log(1 + \sqrt{2})] \), then
\[
\left[ \frac{1}{2} A(a, b) + \frac{1}{2} T(a, b) \right] - [\lambda G(a, b) + (1 - \lambda) T(a, b)] = b \tan^{-1} \frac{x^2 - 1}{x^2 + 1} F_1(x) \quad (3.28)
\]
and
\[
[\omega A(a, b) + (1 - \omega) T(a, b)] - \left[ \frac{1}{5} G(a, b) + \frac{4}{5} T(a, b) \right] = b \tan^{-1} \frac{x^2 - 1}{x^2 + 1} F_2(x), \quad (3.29)
\]
where
\[
F_1(x) = \left( \frac{x^2}{4} - \lambda x + \frac{1}{4} \right) \tan^{-1} \frac{x^2 - 1}{x^2 + 1} + \frac{2 \lambda - 1}{4} (x^2 - 1) \quad (3.30)
\]
and
\[
F_2(x) = \left[ \frac{\omega(x^2 + 1) - x}{2} - \frac{1}{5} \right] \tan^{-1} \frac{x^2 - 1}{x^2 + 1} - \frac{\omega(x^2 - 1)}{2} + \frac{x^2 - 1}{10}, \quad (3.31)
\]
respectively. Simple computations yield
\[
F_1(1) = F_1'(1) = F_1''(1) = 0, \quad F_1'''(1) = 5 \lambda - 1 = -0.4555 \cdots < 0, \quad (3.32)
\]
\[
\lim_{x \to +\infty} F_1(x) = \lim_{t \to 0^+} \left( \frac{t^2 - 4 t + 1}{4 t^2} + (1 - 2 \lambda)(t^2 - 1) \right) = +\infty, \quad (3.33)
\]
\[
F_2(1) = F_2'(1) = F_2''(1) = 0, \quad F_2'''(1) = 1 - 2 \omega = -0.0148 \cdots < 0, \quad (3.34)
\]
and
\[
\lim_{x \to +\infty} F_2(x) = \lim_{t \to 0^+} \left[ \frac{5 \omega(t^2 + 1) - 2 t}{10 t^2} \tan^{-1} \frac{1 - t^2}{t^2 + 1} + (1 - 5 \omega)(1 - t^2) \right] = +\infty. \quad (3.35)
\]
Equations (3.28), (3.32) and (3.33) imply that there exist small enough $\delta_1 > 0$ and large enough $X_1 > 0$ such that $\frac{1}{2} A(a, b) + \frac{1}{2} T(a, b) < \lambda G(a, b) + (1 - \lambda) T(a, b)$ for $\sqrt{a/b} \in (1, 1 + \delta_1)$, and $\frac{1}{2} A(a, b) + \frac{1}{2} T(a, b) > \lambda G(a, b) + (1 - \lambda) T(a, b)$ for $\sqrt{a/b} \in (X_1, +\infty)$.

Equations (3.29), (3.34) and (3.35) imply that there exist small enough $\delta_2 > 0$ and large enough $X_2 > 0$ such that $\omega A(a, b) + (1 - \omega) T(a, b) < \frac{1}{5} G(a, b) + \frac{4}{5} T(a, b)$ for $\sqrt{a/b} \in (1, 1 + \delta_2)$, and $\omega A(a, b) + (1 - \omega) T(a, b) > \frac{1}{5} G(a, b) + \frac{4}{5} T(a, b)$ for $\sqrt{a/b} \in (X_2, +\infty)$.

References