A simple Markov chain for the extended Collatz problem

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Abstract

The paper deals with the Collatz problem. A simple extension of the classic $3n+1$ version is considered allowing the definition of the algorithm for any *target* base. Then a simple three "states" Markov chain is built up to show the *probabilistic* convergence of the algorithm to the equilibrium point.

Keywords: Collatz Problem, Dynamic System, Markov Chain

* Matteo Aicardi is the son of Michele Aicardi. He has suggested the extension of the Collatz problem reported in the paper.

1 Introduction

The Collatz conjecture/problem, also known as the $3n+1$ conjecture or Hasse's problem or Syracuse problem, is one of the many unproven statements in the number theory. Starting form the original statement regarding the finite time attainment of recursive computation

$$n \longleftarrow \begin{cases} 1 & \text{if } n=1, \\ \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases} \tag{1.1}$$

to $n=1$, a great number of studies and extensions have been done [1], [3], [5], [4]. An excellent survey on the bibliography is given by [2].

In this paper, we propose a very simple representation of the problem extended to an arbitrary base, that despite its simplicity, is sufficient to give a *probabilistic* proof of the above conjectured behavior.

2 The associated dynamic system and its probabilistic convergence

2.1 The dynamic system

Definition 2.1. Let the mapping underlying the extended Collatz conjecture be represented by the following dynamic system:

$$n_{i+1} = \begin{cases} 1 & \text{if } n_i = 1, \\ \frac{n_i}{d} & \text{if } n_i \in \mathbb{N} \text{ is a multiple of } d, \\ f(d)n_i + g(d) & \text{if } n_i \in \mathbb{N} \text{ is not a multiple of } d \end{cases} \tag{2.1}$$

with $f(d)$ and $g(d)$ such that:

$$\frac{f(d)n_i + g(d)}{d} \in \mathbb{N} \tag{2.2}$$

$$\frac{d^k - g(d)}{f(d)} \in \mathbb{N} \quad \forall k \in S \subseteq \mathbb{N} \text{ and } \mathrm{card}(S) = \aleph_0 \tag{2.3}$$

△

Condition (2.3) assures that an infinite number of n exist generating a power of d through the third of (2.1).

Remark. The original Collatz problem is obviously related with $d = 2$, $f(d) = 3$ and $g(d) = 1$. Condition (2.3) is assured with $S = \{k = 2z,\ z \in \mathbb{N}\}$.

□

2.2 The Markov Chain and its limiting probability

The aim will be now that of proving that, for any starting point n_0, the trajectories of (2.1) will converge to $n = 1$ with probability 1

To do this, let partition $\mathbb{N}$ in three classes:

- $C_1 : \{n \in \mathbb{N}$ such that $n = d^k$ for some $k \in \mathbb{N} \cup \{0\}\}$ (powers of d);
- $C_2 \quad \{n \in \mathbb{N}$ such that $n = d^p z$ for some $p \in \mathbb{N}$ and odd $z \in \mathbb{N}\}$ (multiple but not powers of d);
- $C_3 : \{n \in \mathbb{N}$ such that $\frac{n}{d} \notin \mathbb{N}\ \}$ (not multiple of d);

Then, consider the following Markov chain:

$$P_{i+1} = MP_i \tag{2.4}$$

where P_i is a vector with three components each of which represents the probability of n_i to belong to one of the above defined classes. I.e. $P_i(1) = \text{Prob}\{n_i \in C_1\}$. M is a $3x3$ real matrix whose elements, say for instance $m_{h,k}$, are the probability that n_{i+1} will belong to class C_h given that $n_i \in C_k$. Such elements are the "transition probabilities".

Then, on the basis of (2.1) we can state that:

1. if $n_i \in C_1$ then $n_{i+1} \in C_1$. This means $m_{1,1} = 1$, $m_{2,1} = m_{3,1} = 0$;
2. if $n_i \in C_2$ then n_{i+1} may either stay in C_2 or make a transition to C_3. Let the probabilities of such transitions be p and $(1 \quad p)$ respectively. As to the elements of matrix M this implies $m_{2,2} = p$ and $m_{3,2} = (1 \quad p)$. Later it will be apparent that the actual value of p does not matter provided that $0 < p < 1$;
3. if $n_i \in C_3$ then n_{i+1} may either go to C_1 or C_2. Denote with q and $(1 \quad q)$ such transition probabilities. This means $m_{1,3} = q$ and $m_{2,3} = (1 \quad q)$. Hereagain the actual value of q does not matter provided that $0 < q < 1$. Condition (2.3) assures that $q \neq 0$.

The Markov chain becomes:

$$P_{i+1} = \begin{bmatrix} 1 & 0 & q \\ 0 & p & (1 \quad q) \\ 0 & (1 \quad p) & 0 \end{bmatrix} P_i \tag{2.5}$$

2.3 The probabilistic convergence the dynamic system

The limiting probabilities of (2.5), i.e.

$$\lim_{i \to \infty} P_i \tag{2.6}$$

can be found determining the solution(s) of the problem:

Problem 2.2. *Find P such that $P = MP$, $P(\cdot) > 0$ and $P(1) + P(2) + P(3) = 1$.*

▲

Now,the Markov chain described by (2.5) exhibits (for p and q different from 0 and 1) a simple eigenvalue in $\lambda = 1$ that implies

$$eig(I \quad M) = \{0, \lambda_1 \neq 0, \lambda_2 \neq 0\} \tag{2.7}$$

Then, there exist an unique solution to the above problem, that is:

$$P(1) = 1 \quad , \quad P(2) = P(3) = 0 \tag{2.8}$$

The matrix structure which yelds to the above conditions, show that the Markov chain has an *absorbing state* that eventually will characterize the trajectory. The computing procedure (if seen as a stochastic process) will eventually generate values of n belonging to C_1. Once entered such a class, n will stay forever there (with a equilibrium point defined by $n = 1$). Then, we can state the result of the paper through the following:

Theorem 2.3. *The dynamic system (2.1), if seen as a stochastic process, has trajectories that, irrespectively of the initial state, converge with probability 1 to the value $n = 1$*

■

3 Conclusions

In the paper it has been shown that, using a simple structured Markov chain, the procedure 2.1 representing the extended Collatz computation, converges to 1 with probability 1, provided the elements d, $f(d)$ and $g(d)$ satisfy (2.2) and (2.3).

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