CONNECTED DOMINATING SET IN INTERVAL-VALUED FUZZY GRAPHS

Faisal M. AL-Ahmadi1* and Mahiub M. Q. Shubatah2

1Department of Mathematics, Sheba Region University, Faculty of Education, Arts and Science Marib, Yemen
2Department of Mathematics, Faculty of Science and Education, Albaydha University, Albaydha, Yemen

*E-mail: faissalalahmadi@gmail.com. 2E-mail: mahioub70@yahoo.com

Abstract: - The connected domination is one of the important issues in mathematics and other science. In This paper, connected domination in interval-valued fuzzy graphs is defined and studied. Some bounds on connected domination number \(\gamma_c(G)\) is provided for several interval-valued fuzzy graphs. Furthermore, the relationship of \(\gamma_c(G)\) with some other known parameters in interval-valued fuzzy graphs are investigated with some suitable examples.

Keywords: - Interval-valued fuzzy graph, connected dominating set, connected domination number.

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1 INTRODUCTION
Rosenfeld [7] in (1975) introduced the nation of fuzzy graphs and widely definitions for each fuzzy vertex, fuzzy edges and several fuzzy analogs of graph theoretic concepts like paths, cycles, connectedness etc. Zadeh in (1975)[13] introduced another detailed the idea of interval-valued fuzzy sets as an extension of fuzzy sets, which gives more precise tool to model uncertainty in real-life situations. Interval-valued fuzzy sets have been widely used in many areas of science and engineering. In (2011) Akram and Dudek [1] introduced the notion of an interval-valued fuzzy graph and studied several properties and operations on interval-valued fuzzy graphs. In (1979) the Connected Domination in graphs was dis- cussed by Sampathkumar and Walikar [8]. The concept of domination in fuzzy graphs was investigated by A. SomaSundaram and S. SomaSundaram [11] further A. SomaSundaram presented the concepts of independent domination, total domination, connected domination of fuzzy graphs [12]. The concept of connected domination in fuzzy graphs using strong arcs was introduced by Manjusha and Sumitha in (2015) [5]. Sarala and Janaki in (2019) [9] discussed the concept of connected domination number in fuzzy digraphs. The concepts of domination in interval-valued fuzzy graphs was investigated by Pradip Debnath in (2013) [3], Because of the Large area for dominating applications in the real-life, like computer networks and the Internets network. In this paper we introduce and study the concepts of connected domination number in interval-valued fuzzy graphs. We obtain many results related to these concepts. Finally the relationship between these concepts and the others in interval-valued fuzzy graph will be obtained with some examples.

2 Preliminaries
In this section we will review some basic concepts that have a relationship to our subject.

Definition 2.1. [7] Let \( G = (V, \mu, \rho) \) is called a fuzzy graph of crisp graph \( G' = (\mu', \rho') \), where \( V \) is a vertex set of \( G, \mu \) is a fuzzy subset of \( V \) and \( \rho \) is a fuzzy relation on \( \mu \) or \( \rho \) the fuzzy edge set of \( G \) and \( \rho(u,v) \leq \mu(v) \land \mu(v) \) for \( u \) and \( v \) in \( V \).

Definition 2.2. [13] A subset \( A \) of a vertex set \( V \) is called interval valued fuzzy set and it is denoted by \( A = [\mu(u), \mu_2(u)] : u \in V \), where the function \( \mu_1 : V \rightarrow [0, 1] \) and \( \mu_2 : V \rightarrow [0, 1] \), such that \( \mu_1(u) \leq \mu_2(u) \) for \( u \in V \). If \( G^* = (V, E) \) is a crisp graph, then by an interval valued fuzzy relation \( \rho = (\rho_1, \rho_2) \) on \( V \), we mean an interval valued fuzzy on \( E \), such that and

\[
\rho_1(u,v) \leq \mu_1(u) \land \mu_1(v)
\]

\[
\rho_2(u,v) \leq \mu_2(u) \land \mu_2(v)
\]

for all \( (u,v) \in E \) and denoted by \( B = \{(u,v), [\rho_1(u,v), \rho_2(u,v)] : (u,v) \in E \} \), where the function \( \rho_1 : E \rightarrow [0, 1] \) and \( \rho_2 : E \rightarrow [0, 1] \), such that \( \rho_1(u,v) \leq \rho_2(u,v) \) for \( (u,v) \in E \).

Definition 2.3. [1, 4] An interval-valued fuzzy graph of the graph \( G^* = (V, E) \) is a pair \( G = (A, B) \) where

1. \( A = [\mu_1, \mu_2] \) is an interval-valued fuzzy set on \( V \).
2. \( B = [\rho_1, \rho_2] \) is an interval-valued fuzzy relation on \( V \), such that

\[
\rho_1(u,v) \leq \min\{\mu_1(u), \mu_1(v)\}
\]

and for all

\[
(u,v) \in E,
\]

\[
\rho_2(u,v) \leq \min\{\mu_2(u), \mu_2(v)\}
\]
Definition 2.4. [1] An edge $e = (x, y)$ of an interval-valued fuzzy graph is called effective edge if $\rho(x, y) = \min [\mu(x), \mu(y)] = \min [\mu(x), \mu(y)]$. The degree of a vertex can be generalized in different ways for an interval-valued fuzzy graph $G = (A, B)$ of $G^* = (V, E)$. The effective degree of a vertex $v$ in an interval-valued fuzzy graph $G = (A, B)$ is defined to be sum of the weights of the effective edges incident at $v$ and it is denoted by $d_{\rho}(v)$. The minimum effective edges degree of $G$ is $d_{\rho}(G) = \min \{ d_{\rho}(v)\vert v \in V \}$. The maximum effective degree of $G$ is $\Delta_{\rho}(G) = \max \{ d_{\rho}(v)\vert v \in V \}$.

Definition 2.5. [1] Two vertices $u$ and $v$ are said to be neighbors in interval-valued fuzzy graph $G = (A, B)$ of $G^* = (V, E)$ if $\rho(u, v) = \mu(u) \land \mu(v)$ and $\rho(u, v) = \mu(u) \lor \mu(v)$. A vertex subset $N(v) = \{ u \in V : v \text{ adjacent to } u \}$ is called the open neighborhood set of a vertex $v$ and $N[v] = N(v) \cup \{ v \}$ is called the closed neighborhood set of $v$. The neighborhood degree of a vertex $v$ in an interval-valued fuzzy graph $G = (A, B)$ is defined to be the sum of the weights of the vertices adjacent to $v$ and it is denoted by $d_{\rho}(v)$, that is $d_{\rho}(v) = \rho(v)$. The minimum neighborhood degree of $G$ is $d_{\rho}(G) = \min \{ d_{\rho}(v)\vert v \in V \}$. The maximum neighborhood degree of $G$ is $\Delta_{\rho}(G) = \max \{ d_{\rho}(v)\vert v \in V \}$.

Definition 2.6. [6] An interval-valued fuzzy graph $G = (A, B)$ is said to be complete interval-valued fuzzy graph if $\rho(v, u) = \min [\mu(v), \mu(u)]$ and $\rho(v, u) = \min [\mu(u), \mu(v)]$ for all $v, u \in V$ and is denoted by $K_{\rho}$. The complement of an interval-valued fuzzy graph $G = (A, B)$ is an interval-valued fuzzy graph $G = (V, E)$ where

(ii) $\mu_1 \leq \mu_2$ for all vertices; and

(iii) $\rho(u, v) = \min [\mu(u), \mu(v)] - \rho(u, v)$ and $\rho(v, u) = \min [\mu(v), \mu(u)] - \rho(v, u)$ for all $u, v \in E$.

Definition 2.7. [6] An interval-valued fuzzy graph $G = (A, B)$ is said to be bipartite if the vertex set $V$ of $G$ can be partitioned into two non empty sub sets $V_1$ and $V_2$ such that $V_1 \cap V_2 = \emptyset$. A bipartite interval-valued fuzzy graph $G = (A, B)$ is said to be complete bipartite interval-valued fuzzy graph if $\rho(v_i, v_j) = \min [\mu(v_i), \mu(v_j)]$ and $\rho(v_i, v_j) = \min [\mu(v_j), \mu(v_i)]$ for all $v_i \in V_1$ and $v_j \in V_2$. It is denoted by $K_{\rho,n}$, where $|V_1| = m, |V_2| = n$.

Definition 2.8. [3] A vertex $u \in V$ of an interval-valued fuzzy graph $G = (A, B)$ is said to be an isolated vertex if $\rho(v, u) = 0$ and $\rho(v, u) = 0$ for all $v \in V$. That is, $N(u) = \emptyset$.

Definition 2.9. [10] An interval-valued fuzzy graph $G = (A, B)$ is said to be strong interval-valued fuzzy graph, if $\rho(u, v) = \min [\mu(u), \mu(v)]$ and $\rho(u, v) = \min [\mu(u), \mu(v)]$ for all $(u, v) \in E$.

Definition 2.10. [1] An interval-valued fuzzy graph $H = (A, B_1)$ is said to be an interval-valued fuzzy sub graph of $G = (A, B)$, if $A_1 \subseteq A$ and $B_1 \subseteq B$. That is, $\mu' \leq \mu$, $\mu' \leq \mu$ and $\rho' \leq \rho$.

Definition 2.11. [3] Let $G = (A, B)$ be an interval-valued fuzzy graph and let $S \subseteq V(G)$. Then a vertex subset $S$ of $G$ is said to be independent set if $\rho(u, v) < \mu_1(u) \cap \mu_1(v)$ and $\rho(u, v) < \mu_2(u) \lor \mu_2(v) \lor \mu_2(v)$ or $\rho(u, v) = 0$ and $\rho(u, v) = 0$ for all $u, v \in S$. An independent vertex subset $S$ of an interval valued fuzzy graph $G = (A, B)$ is said to be maximal independent set if $S \cup \{ u \}$ is not independent $\forall u \in V - S$. The maximum fuzzy cardinality among all maximal independent sets in interval valued fuzzy graph $G$ is called the independence number of $G$ and is denoted by $\beta_G$.

Definition 2.12. [3] Let $G = (A, B)$ be an interval-valued fuzzy graph and let $u, v \in V(G)$. Then we say that $u$ dominates $v$ or $v$ dominates $u$ if $(u, v)$ is an effective edge, i.e $\rho(u, v) = \min [\mu(u), \mu(v)]$ and $\rho(u, v) = \min [\mu(u), \mu(v)]$. A vertex subset $D \subseteq V(G)$ is called dominating set in an interval valued fuzzy graph $G$, if for every $v \in V - D$ there exists $u \in D$ such that $(u, v)$ is effective edge. A dominating set $D$ of an interval valued fuzzy graph $G$ is called minimal dominating set if $D - \{ u \}$ is not dominating set for every $u \in D$. A minimal dominating set $D$ with $|D| = \gamma(G)$ is denoted by $\gamma - set$.

3 Connected Domination Of An Interval-Valued Fuzzy Graph.

Definition 3.1. Let $G = (A, B)$ be interval valued fuzzy sub graph of graph $G^* = (V, E)$ and let $v \in V$. Then $G$ is a connected interval valued fuzzy graph if for every $v; u \in V(G)$. Thence is a path $u - v$ spanning $u$ and $v$.

Definition 3.2. A dominating set $D$ in an interval valued fuzzy graph $G$ is said to be connected dominating set if the induced interval valued fuzzy sub graph $\langle D \rangle$ is connected and is denoted by $D_c$.

Definition 3.3. A connected dominating set $D_c$ of an interval valued fuzzy graph $G$ is said to be minimal connected dominating set if for every $u \in D_c$, $D_c - \{ u \}$ is not connected dominating set of $G$. 
Definition 3.4. The minimum fuzzy cardinality of all minimal connected dominating set of interval-valued fuzzy graph \( G \) is called the connected domination number of \( G \) and denoted by \( \gamma_c(G) \).

Definition 3.5. The maximum fuzzy cardinality of all minimal connected dominating set of interval-valued fuzzy graph \( G \) is called the upper connected domination number of \( G \) and denoted by \( \Gamma_c(G) \).

Remark 3.6. A minimal connected dominating set \( D_c \) of an interval-valued fuzzy graph \( G \), with \( |D_c| = \gamma_c(G) \) is the minimal connected dominating set of \( G \) and is denoted by \( \gamma_c \)-set.

Example 3.7. For the interval-valued fuzzy graph \( G \) given in Figure 1, such that all edges of \( G \) are effective.

We see that, \( D_{c1} = \{v_2, v_4\} \) and \( D_{c2} = \{v_3, v_4\} \) are minimal connected dominating sets.

Since \( D_{c1} = \{v_2, v_4\} \) is a \( \gamma_c \)-set of \( G \), then \( \gamma_c(G) = 1.3 \). And \( D_{c2} = \{v_3, v_4\} \) is the maximum connected dominating set of \( G \), then \( \Gamma_c(G) = 1.4 \).

Theorem 3.8. Every connected dominating set of an interval-valued fuzzy graph \( G \) is a dominating set of \( G \). But the converse is need not be true.

Proof. Let \( G = (A, B) \) be an interval-valued fuzzy graph and let \( D_c \) be a connected dominating set of \( G \) by definition of the connected dominating set of an interval-valued fuzzy graph \( G \), \( D_c \) is dominating set of \( G \).

The converse of the above theorem need not be true.

For the interval-valued fuzzy graph given in Figure 1, we have \( D = \{v_1, v_3\} \) is dominating set of \( G \), but it is not connected.

Corollary 3.9. For any interval-valued fuzzy graph \( G \), \( \gamma(G) \leq \gamma_c(G) \).

Proof. Since every connected dominating set of an interval-valued fuzzy graph \( G \) is dominated set of \( G \), by the above Theorem. Hence \( \gamma(G) \leq \gamma_c(G) \).

Theorem 3.10. Let \( G = (A, B) = K_{n} \) be any complete interval-valued fuzzy graph and let \( D_c \) be a connected dominating set of \( G \). Then \( V - D_c \) is a connected dominating set of \( k_n \).

Proof. Let \( G = K_{n} \) be a complete interval-valued fuzzy graph and let \( D_c \) be a connected dominating set of \( G \). Then all edges in \( G \) are strong edges and each vertex in \( G \) dominates to all the other vertices of \( G \). Hence a connected dominating set \( D_c \) contain one vertex say \( v \). Since all edges in \( G \) are strong arcs and each vertex in \( D_c \) dominates to all the other vertices in \( G \). Therefore \( V - D_c = V - \{v\} \) is also a connected dominating set of \( G \).

Theorem 3.11. Let \( G \) be any complete interval-valued fuzzy graph and let \( G \) be the complementary of \( G \). Then connected dominating set in \( G \) is not exists.

Proof. Let \( G \) be any complete interval-valued fuzzy graph and let \( G \) be a complement of \( G \). Then all edges of \( G \) are strong edges and every vertex in \( G \) dominates to all the other vertices of \( G \). By the above Theorem, connected dominating set exists. Since \( G \) is the complementary of \( G \). Then \( G \) is null interval-valued fuzzy graph. Therefore the connected dominating set of \( G \) is not exists.

Corollary 3.12. For any complete interval-valued fuzzy graph \( G = K_{n} \). Then

\[ \gamma(G) = \gamma_c(G). \]

Theorem 3.13. A connected dominating set \( D_c \) of interval-valued fuzzy graph \( G \) is minimal connected dominating set if and only if one of the following condition holds:

(i) \( N(v) \cup D_c = \emptyset \)

(ii) There is a vertex \( u \in V - D_c \) such that \( N(u) \cap D_c = \{v\} \).

Proof. Let \( G \) be interval-valued fuzzy graph with \( D_c \) is \( \gamma \)-set of \( G \) and let \( v \in D_c \). Then \( D_c - \{v\} \) is not a connected dominating set of \( G \) and there exists a vertex \( u \in V - \{D_c - \{v\}\} \), such that \( u \) is not dominated by any vertex of \( D_c - \{v\} \). If \( u = v \), then \( N(v) \cap D_c = \emptyset \). If \( u \neq v \), then \( N(u) \cap D_c = \{v\} \).
**Conversely:** suppose that \( D_c \) is a connected dominating set and for each a vertex \( v \in D_c \), one of two conditions holds. Suppose that \( D_c \) is not minimal connected dominating set of \( G \), then there exists a vertex \( v \in D_c \), such that \( D_c - \{v\} \) is a connected dominating set of \( G \). Hence \( v \) is adjacent to at least one vertex in \( D_c - \{v\} \), then the condition one does not hold, which a contradiction. If \( D_c - \{v\} \) is connected dominating set, then every vertex in \( V - D_c \) is adjacent to at least one vertex in \( D_c - \{v\} \), then the condition two dose not hold, which a contradiction. So \( D_c \) is minimal connected domanating set of interval-valued fuzzy graph \( G \).

**Theorem 3.14.** Let \( G = (A, B) \) be any connected interval-valued fuzzy graph of \( G^* = (V, E) \) and \( G_1 = (A_1, B_1) \) be any maximum spanning tree of \( G \). Then every connected dominating set of \( G_1 \) is also a connected dominating set of \( G \) and \( \gamma_c(G) \leq \gamma_c(G_1) \).

**Proof.** Let \( D_c \) be a connected dominating set of a maximum spanning tree \( G_1 \) of an interval- valued fuzzy graph \( G \). Since \( G_1 \) is maximum spanning tree of an interval-valued fuzzy graph \( G \) with a vertex set \( V(G_1) = V(G) \). Then for every vertex \( v \in V - D_c \), there exists a vertex \( u \in D_c \) such that \( u \) dominates \( v \). Hence \( D_c \) is a connected dominating set of \( G \). Hence \( \gamma_c(G) \leq \gamma_c(G_1) \).

**Example 3.15.** For the interval-valued fuzzy graph \( G \) and its maximum spanning interval- valued fuzzy sub graph \( G_1 \) in the following two Figures, such that each edges of \( G \) and \( G_1 \) are strong edges.

![Fig 2](image1.png)

We see that, \( D_c(G) = \{v_3, v_4\} \) is a connected dominating set of an interval-valued fuzzy graph \( G \). Then a connected domination number of \( G \), \( \gamma_c(G) = 1.4 \).

Also we see that, \( D_c(G_1) = \{v_2, v_3, v_4, v_5\} \) is a connected dominatig set of an interval-valued fuzzy sub graph \( G_1 \). Then a connected domination number of \( G_1 \), \( \gamma_c(G_1) = 2.5 \).

Hence \( \gamma_c(G) \leq \gamma_c(G_1) \).

**Remark 3.16.** Since a connected dominating set \( D_c = n - 2 \) for any strong path \( P_n \), strong cycle \( C_n \) and bipartite \( K_{n_1,n_2} \) interval-valued fuzzy graph \( G \) with \( n \geq 3 \), where \( n \) the number of vertices of \( G \) and \( n_1 + n_2 = n \).

**Theorem 3.17.** For any strong path \( P_n \) interval-valued fuzzy graph \( G \) with \( n \geq 3 \). Then

\[
|D'| = \overline{\phi}(G) = \sum_{n=3}^{\infty} |D'|
\]

for \( v_i \in V \).

**Proof.** Let \( G = P_n \) be a strong path interval-valued fuzzy graph with \( n \geq 3 \), where \( n \) the number of vertices of \( G \). Since a path \( P \) of length \( n \) is a sequence of vertices \( v_0, v_1, \ldots, v_n \).

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where \( \mu(v_{i-1}, v_i) > 0 \), \( i = 1, 2, ..., n \). Then connected dominating set \( D_c = n - 2 \) by the above Remark, i.e \( D_c = \{v_1, v_2, ..., v_{n-1}\} \) except \( v_n \). Hence \( |D_c| = |v_1| + ... + |v_{n-1}| = \sum_{i=1}^{n-1} \mu_i \).

Therefore

\[
|D_c| = \gamma_c(G) = \sum_{i=1}^{n-2} |v_i|
\]

**Example 3.18.** For interval-valued fuzzy graph given in Figure (4), such that all edges of \( G \) are strong.

\[
\begin{array}{c}
|v_1| [0.2, 0.5] & |v_2| [0.1, 0.2] & |v_3| [0.3, 0.7] & |v_4| [0.3, 0.6] & |v_5| [0.5, 0.5] \\
G: & Fig 4 & \end{array}
\]

From the above Figure, we have \( D_c = \{v_2, v_3, v_4\} \) is connected dominating set of \( G \). Then connected domination number

\[
|D_c| = \sum_{i=1}^{n-2} |v_i| = \sum_{i=1}^{3} |v_i|
\]

\[
= |v_2| + |v_3| + |v_4| = 1.9
\]

**Theorem 3.19.** For any bipartite \( K_{n_1, n_2} \) interval-valued fuzzy graph \( G \) with \( n \geq 3 \). Then

\[
|D_c| = \gamma_c(G) = \sum_{i=1}^{n-2} |v_i|,
\]

for \( v_i \in V \).

**Proof.** Let \( G = K_{n_1, n_2} \) be bipartite interval-valued fuzzy graph with \( n \geq 3 \), where \( n = n_1 + n_2 \) such that \( n_1 \) and \( n_2 \) the number of vertices \( V_1 \) and \( V_2 \) of \( G \) respectively. Since a vertex set \( V_1 \) dominates a vertex set \( V_2 \) and conversely any bipartite which it is independent dominating set of \( G \), but it is not connected dominating set of \( G \). Hence connected dominating set \( D_c \) in the bipartite is the same connected dominating set in the path. By the above Theorem, then

\[
|D_c| = \gamma_c(G) = \sum_{i=1}^{n-2} |v_i|
\]
Example 3.20. Consider the interval-valued fuzzy graph $G$, shown in Figure (5) such that all edges of $G$ are effective.

From the Figure, we see that $D_c = \{v_2, v_3, v_4, v_5\}$ is a connected dominating set of $G$. Since

$$|D_c| = \sum_{i=1}^{n-2} |v_i|,$$

for $v_i \in V$. Then

$$|D_c| = \gamma_c(G) = \sum_{i=1}^{4} |v_i|$$

$$= |v_2| + |v_3| + |v_4| + |v_5| = |0.8 + 0.6 + 0.6 + 0.75| = 2.75$$

Remark 3.21. For any bipartite, a strong cycle or a strong path interval-valued fuzzy graph $G$ with $n \geq 4$, where $n$ the number of vertices of $G$. Then

(1) $\gamma_c(G) \geq p - \Delta_N(G)$, if $\mu_1(u) = \mu_2(u)$ to each $u \in V - D_c$, and $\mu_1(v) \leq \mu_2(v)$ to each $v \in D_c$.

(2) $\gamma_c(G) \leq p - \Delta_N(G)$, if $\mu_1(u) \leq \mu_2(u)$ to each $u \in V - D_c$ and $\mu_1(v) = \mu_2(v)$ to each $v \in D_c$.

(3) $\gamma_c(G) + \frac{1}{2} \geq p - \Delta_N(G)$ for any interval-valued fuzzy graph $G$, namely $\mu_1(u) \leq \mu_2(u)$ to each $u \in V$. 

Theorem 3.22. For any bipartite interval-valued fuzzy graph $G = K_{n_1,n_2}$ with $n \geq 4$, $n = n_1 + n_2$, where $n_1$ and $n_2$ the number of vertices $V_1$ and $V_2$ respectively. Then

$$\frac{p}{\Delta_N(G) + 1} \leq \gamma_c(G)$$

Proof. Let $G$ be bipartite interval-valued fuzzy graph with $n \geq 4$, $n = n_1 + n_2$, where $n_1$ and $n_2$ the number of vertices $V_1$ and $V_2$ respectively with $\gamma_c$ - set and let $v$ be a vertex of $G$ with $\Delta_N(v) = d_N(v)$. Since $\gamma_c(G) \leq p - \Delta_N(G)$ or $\gamma_c(G) \geq p - \Delta_N(G)$. Then $p - \gamma_c(G) \leq \Delta_N(G)$ or $p - \gamma_c(G) \geq \Delta_N(G)$. Thus clear that

$$p - \gamma_c(G) \leq \gamma_c(G) - \Delta_N(G).$$

$$p \leq \gamma_c(G) - \Delta_N(G) + \gamma_c(G)$$

$$p \leq \gamma_c(G) - \Delta_N(G) + 1$$

$$\frac{p}{\Delta_N(G) + 1} \leq \gamma_c(G)$$

Example 3.23. From the above figure (5), we have $p = 8.2$, $\gamma_c(G) = 6.85$ and $\Delta_N(G) = 1.45$. Then
\[
\frac{p}{\Delta_N(G) + 1} \leq \gamma_c(G)
\]
\[
\frac{8.2}{1.45 + 1} \leq 6.85
\]

3.3469 < 6.85

**Theorem 3.24.** For any complete bipartite interval-valued fuzzy graph, \( G = k_{m,n} \). Then
\[
\gamma_c(G) = \min \{|x| : x \in V_1\} + \min \{|y| : y \in V_2\}.
\]

**Proof.** Let \( G = k_{m,n} \) be complete bipartite interval-valued fuzzy graph, where \( m \) and \( n \) are the number of vertices \( V_1 \) and \( V_2 \) respectively. Then all edges in \( G \) are effective and each vertex in \( V_1 \) dominates to all vertices in \( V_2 \) and each vertex in \( V_2 \) dominates to all vertices in \( V_1 \). Hence a connected dominating set of \( G \) contain exactly two vertices \( x, y \) where \( x \in V_1 \) and \( y \in V_2 \). Hence \( x \) has the minimum membership value in \( V_1 \) and \( y \) has the minimum membership value of \( V_2 \). Therefore,
\[
\gamma_c(G) = \min \{|x| : x \in V_1\} + \min \{|y| : y \in V_2\}.
\]

\( \square \)

**Example 3.25.** Consider the interval-valued fuzzy graph \( G \), shown in Figure (6), such that all edges of \( G \) are effective.

![Figure 6](image)

From the Figure, we see that a vertex subset \( \{a, b\}, \{a, f\}, \{c, b\}, \{c, f\}, \{d, b\} \) and \( \{d, f\} \) are minimal connected dominating set of an interval-valued fuzzy graph \( G \). Then minimum connected dominating set of \( G = \{a, b\} \). Hence \( \gamma_c(G) = 1.15 \)

**Remark 3.26.** For any complete bipartite interval-valued fuzzy graph \( G = k_{m,n} \) with \( \gamma_c \)-set. Then \( V - D_c \) is a connected dominating set of \( k_{m,n} \).

**Example 3.27.** From the above Example in figure (6), since \( D_c = \{a, b\} \) is \( \gamma_c \)-set. Then \( V - D_c = \{c, f, d\} \) is a connected dominating set of \( G \).
4 Conclusion

In this paper, the concepts of connected dominating set and connected domination number are defined on interval-valued fuzzy graphs and also applied for the various types of interval-valued fuzzy graphs and suitable examples have given. Upper and lower bounds of $\gamma_c(G)$ in interval-valued fuzzy graphs $G$ are obtained. We have done some results with examples and Relations of connected domination number with known parameters in interval-valued fuzzy graph were discussed with the suitable examples.

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