NUMBER OF LEVEL CROSSINGS 
OF RANDOM ALGEBRAIC POLYNOMIALS

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Abstract—In this paper, we have estimated bounds of the number of level crossings of the random algebraic polynomials

\[
 f_n(t,x) = \sum_{k=0}^{n} a_k(t)x^k = 0 
\]

where \( a_k(t) \leq t, 0 \leq t \leq 1 \), are dependent random variables assuming real values only and following the normal distribution with mean zero and joint density function \([M]^{1/2} (2\pi)^{-n/2} \exp[-1/2]\delta M\delta]\). There exists an integer \( n_0 \) and a set \( E \) of measure at most \( A/(\log n_0 - \log \log \log n_0) \) such that, for each \( n > n_0 \) and all not belonging to \( E \), the equations (1.1) satisfying the condition (1.2), have at most \( \alpha (\log \log n)^2 \log n \) roots where \( \alpha \) and \( A \) are constants.

Keywords—Independent identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots.

In this paper we use data science to find the number of zeros and number of level crossings of algebraic polynomial using different methods. Data science often uses statistical inferences to predict or analyze trends from data, while statistical inferences uses probability distributions of data. Hence knowing the probability and its applications are important to work effectively on data science problems and we get the number of zeros of the polynomial we consider, is the most approximate to other predecessors.

I. INTRODUCTION

Consider the family of equations

\[
 f_n(t,x) = \sum_{k=0}^{n} a_k(t)x^k = 0 
\]

where \( a_k(t) \leq t, 0 \leq t \leq 1 \), are dependent random variables assuming real values only and following the normal distribution with mean zero and joint density function.

\[
 [M]^{1/2} (2\pi)^{-n/2} \exp[-1/2]\delta M\delta] 
\]

when \( M^{-1} \) is the moment matrix with \( \sigma_i=1, \rho_{ij}=\rho, 0 < \rho, i \neq j, i, j = 0,1,...,n \) and \( \delta' \) is the transpose of the column vector \( \delta \).

In this paper we estimate the upper bound of the number of real roots of (1.1). We prove the following theorem.

THEOREM: There exists an integer \( n_0 \) and a set \( E \) of measure at most \( A/(\log n_0 - \log \log \log n_0) \) such that, for each \( n > n_0 \) and all not belonging to \( E \), the equations
satisfying the condition (1.2), have at most
\[ \alpha(\log \log n)^2 \log n \] roots where \( \alpha \) and \( A \) are constants.

The transformation \( x \rightarrow \frac{1}{x} \) makes the equation
\[ f_n(x, t) = 0 \] transformed to
\[ \sum_{j=0}^{n} a_{n-j}(t)x^j = 0 \] and \( (a_0(t), a_{n-1}(t), \ldots, a_n(t)) \) have
the same joint density function. Therefore number of roots and the measure of the exceptional set in
the set \([-\infty, -\infty]\) are twice the corresponding value can be considered and now show that this upper
bound is same as in \([0, 1]\).

There are many known asymptotic estimates for
the number of real zeros that an algebraic or
trigonometric polynomial are expected to have
when their coefficients are real random variables.
The present paper considers the case where the
coefficients are complex. The coefficients are
assumed to be independent normally distributed
with mean zero. A general formula for the case of
any complex non stationary random process is also
presented.

Some years ago Kac (1943) gave an asymptotic
estimate for the expected number of real zeros of
an algebraic polynomial where the coefficients are
real independent normally distributed random
variables. Later Ibragimov and Maslova (1971)
obtained the same asymptotic estimate for a case
which included the results due to Kac(1943, 1949),
Littlewood and Offord (1939) and others. They
considered the case when the coefficients belong
to the domain of attraction of normal law. Recently
there has been some interesting development of the
subject, a general survey of which, together with
references may be found in a book by Bharucha-Reid
and Sambandham (1986). These
generalizations consider different types of
polynomials, see for example Dunnage (1966) or
study the number of level crossings rather than
axis crossings, see Farahmand (1986, 1990).

However, they assume the real valued coefficients
only. Dunnage (1968) considered a wide
distribution for the complex-valued coefficients,
however he only obtained an upper limit for the
number of real zeros. Indeed, the limitation of this
result, being only in the form of an upper bound, is
justified. It is easy to see that for the case of
complex coefficients there can be no analogue of
the asymptotic formula for the expected number of
real zeros. To illustrate this point we use the result
due to Dunnage (1968). Suppose
\[ \sum_{j=0}^{n} \left( x_j + i\beta_j \right) f_j(x) \] has a real root where
\( f_j(x) \) is in the form of \( x^j \) or \( \cos j\theta \) and
\( \alpha_j \) and \( \beta_j \), \( j = 0, 1, \ldots, n \) are sequences of
independent random variables. This implies that
the polynomials
\[ \sum_{j=0}^{n} \alpha_j f_j(x) \] and
\[ \sum_{j=0}^{n} \beta_j f_j(x) \] have a common root and the
elimination of \( f_j(x) \) leads to the equation

\[ \phi(\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n) = 0. \]

Thus the number of roots in the range \([ -\infty, -\infty]\)
and the measure of the exceptional set are eachour times the corresponding estimates for the
range \([0,1]\). Evans has considered the case when the random co-efficients are independent and normal. Our technique of proof is analogous to that of Evans.

\(\xi 2\). We define the circles \(C_0, C_e, C_m \) and \(C_1\) as follows. \(C_0\) with centre at \(z=0\) and radius \(\frac{1}{2}\), \(C_e\) with centre at 
\[
z = \frac{3}{4} - \frac{\log \log n_0}{2n_0}
\]
And of radius 
\[
\frac{1}{4} - \frac{\log \log n_0}{2n_0}
\]
\(C_m\) with centre at \(z=1-2^m\) and of radius 
\[
r_m = \frac{1}{2}(1 - X_m) = 2^{-(m+1)} \text{ for } m = m_0, m_1, \ldots, M
\]
where
\[
m_0 = \left[\frac{\log n_0 - \log \log \log n + \log 3}{\log 2}\right] - 1
\]
and
\[
\log n - \log \log \log n - 1 < \frac{\log n - \log \log \log n}{\log 2}
\]
and
\(C_1\) with centre at \(z = 1\) and radius \(\frac{\log \log n}{n}\).

By Jensen’s theorem the number of zeros of a regular function \(\varphi(z)\) in a circle \(z_0\) and of radius \(r\) does not exceed
\[
\frac{\log n(M / \varphi(z_0))}{\log(R / r)}
\]
where \(M\) is the upper bound of \(\varphi(z)\) in a concentric circle of radius \(R\). We use this theorem to find the number of zeros of \(f_n(z, t)\) in each circle. Summing the number of zeros in each of the circle we get the upper bound of the number of zeros of \(f_n(z, t)\) in the circle.

\(\xi 3\). To estimate the upper bound of the number of zeros of \(f_n(z, t)\) in the circle \(C_0\), we shall use the fact that each \(a_k(t)\) has marginal frequent function.

\[
\frac{1}{\sqrt{2\pi}} e^{-r^2 / 2}
\]
Now if \(\max |a_v| > (n + 1)\) then \(|a_v| > (n + 1)\) for at least one value of \(v \leq n\), so that
\[
P(\max |a_v| > n + 1) \leq \sum_{v=0}^{n} P(|a_v| > n + 1)
\]
\[
= (n+1)(2\pi)^{1/2} \int_{n+1}^{\infty} e^{-t^2} dt
\]
\[
< \frac{1}{\sqrt{2\pi}} e^{-((n+1)/2)^2}
\] (3)

Since \(|f_n(z, t)| \leq (n+1)|z|^n \max |a_v|\), in the circle
\[
|z| = 1 + \frac{2 \log \log n}{n},
\]
We get
\[
f_n(x, t) \leq \left(1 + \frac{2 \log \log n}{n}\right)^n (n+1) \max |a_v|
\]
\[
< (n+1)^2 e^{2 \log \log n}
\] (4)

Outside a set of measure at most
\[
(2/\pi)^{1/2} e^{-(1/2)(n+1)^2}
\]
by (3).
\[
|f_n(0, t)| = |a_0| \text{ and}
\]
\[
P(|a_0| > (n+1)^{-2}) = (2\pi)^{1/2} \int_0^{(n+1)^{-2}} e^{-u^2/2} du < (2\pi)^{1/2} (n+1)^{-2}.
\]
Hence outside a set of measure at most
\[
(2/\pi)^{1/2} (n+1)^{-2}
\] we have
\[ |f_n(0,t) = |a_0(t)| \geq (n+1)^{-2} \]

If \( N_0 \) denotes the number of zeros of \( f_n(z,t) \) in the circle \( C \) then Jensen’s theorem (J), (4) and (5) we have

\[
N_0 < \frac{2\log 2 \log 4(n+1)^2 + 2\log 2 \log n}{\log 2}
\]

Outside a set of measure at most
\[
((2/\pi)^{1/2} e^{-(n+1)/2} + (2/\pi)^{1/2}(n+1)^{-2})
\]

Thus for all \( n > n_0 \), we have

\[
N_0 < \frac{4\log 2 \log n + 2\log 2 \log n}{\log 2}
\]

Outside a set of measure at most

\[
\sum_{n=n_0+1}^{\infty} (2/\pi)^{1/2} e^{-(n+1)/2} + (n+1)^{-1} < C/n_0
\]

Where \( C \) is an absolute constant

4. To estimate the upper bound of the number of zeros of \( f_n(x,t) \) in the circle \( C \), we proceed as follows. The probability that

\[
\sum_{n=1}^{\infty} |g_n|^{2} < (n+1)^{-2}
\]

is

\[
(2/\pi)^{1/2} \int e^{-2\sigma_0^2} du < (2/\pi)^{1/2}(n+1)^{-1}\sigma_0^{-1}
\]

(5)

Where

\[
\sigma_0^{-1} = (1-\rho) \sum_{i=0}^{\infty} \left( \frac{3}{4} - \frac{\log n_i}{2n_0} \right)^{2} + \rho \left( \sum_{i=0}^{\infty} \left( \frac{3}{4} - \frac{\log n_i}{2n_0} \right)^{2} \right)
\]

\[
>(1-\rho)^{-1} \left( 1 - \frac{3}{4} \frac{\log n_0}{2n_0} \right)^{-2}
\]

(6)

If \( N_0 \) denotes the number of zeros of \( f_n(z,t) \) in the circle \( C \) then Jensen’s theorem (J), (4), (5) and (6) we have

\[
N_0 < \frac{4\log 2 \log n + 2\log 2 \log n}{\log 2}
\]

Outside a set measure at most

\[
\sum_{n=n_0+1}^{\infty} \left( \frac{2}{\pi} e^{-(n+1)/2} + \frac{1}{(n+1)^{2}\sigma_n} \right) < \frac{C}{n_0^{1/2}} \left( \frac{\log n_0}{1-(\log n_0)^2} \right)^{-1/2}
\]

5. To obtain an upper estimate of the number of zeros of \( f_n(x,t) \) in the circle \( C \), we need the following lemmas.

**LEMMA 1**: Let \( E \) be an arbitrary set. Then for complex numbers \( g \), we have

\[
\int_{E} \log \left| \sum_{v=0}^{n} a_v(t) g_v \right| dt
\]

\[
< m(E) \log \sigma + m(E) \log \log \left( \frac{C}{m(E)} \right)
\]

\[
\sigma^2 = (1-\rho) \sum_{v=0}^{\infty} |g_v|^2 + \rho \left( \sum_{v=0}^{\infty} |g_v|^2 \right)^2
\]

**PROOF**: Let \( g_v = b_v + ic_v \) where \( b_v \) and \( c_v \) are real. Also let

\[
F = \left\{ t : \sum_{v=0}^{\infty} |a_v(t)g_v| \geq A\sigma \right\}
\]

\[
G = \left\{ t : \sum_{v=0}^{\infty} |a_v(t)b_v| \geq A\sigma / 2^{1/2} \right\}
\]

and

\[
H = \left\{ t : \sum_{v=0}^{\infty} |a_v(t)c_v| \geq A\sigma / 2^{1/2} \right\}
\]
\[ m(G) = \sigma_n^{-1} \left( 2/\pi \right)^{1/8} \int_{A_\theta}^{1/2} e^{1/8} \sigma_n^8 \, du < \frac{2}{A_\theta^{1/2}} e^{1/8} \]

And

\[ m(H) \leq \frac{2}{A \pi^{1/2}} e^{1/8} \]

Since

\[ F \subset G \cup H \quad \text{and} \quad m(F) \leq m(G) + m(H) \leq \frac{4}{A \pi^{1/2}} e - d^{1/4} \]

. Following Evans [Lemma] we get the proof of the lemma.

**LEMMA 2**: If \( g, v = 0,1, \ldots \) are real and if

\[ G = \left\{ t : \sum_{v=0}^{\infty} a_v(t) g_v \leq g \sigma \right\} \]

Then \( m(G) < tQ \), where

\[ \sigma^2 = (1 - \rho) \sum_{v=0}^{\infty} g_v^2 + \rho \left( \sum_{v=0}^{\infty} g_v \right)^2 \]

\[ \sigma_n^2 = (1 - \rho) \sum_{v=0}^{n} g_v^2 + \rho \left( \sum_{v=0}^{n} g_v \right)^2 \]

and

\[ Q = \left( 2/\pi \right)^{1/4} (\sigma / \sigma_n) \]

and if \( E \) is any set having no point in common with \( G \) then

\[ \int_{E} \log \left| \sum_{v=0}^{n} a_v(t) g_v \right| dt > m(E) \log \sigma - CQm(E) \log \frac{1}{m(E)} \]

**PROOF**: Following Evans [Lemma2] we get the proof of the lemma.

Let \( N_m (r, t) \) denote the number of zeros of \( f_v(z, t) \) in the circle with centre \( x_m \) and radius \( r \). By Jensen’s theorem

\[ \delta 1^{n+3} \int_{B}^{r} \frac{N_m(r, t)}{r} \, dr = \frac{1}{2\pi} \int_{|z - m| = t} \log \left| \frac{f_n(z, t)}{f_n(x_m, t)} \right| \, dz \]

Therefore, writing

\[ \varphi_m(t) \text{ for } N_m(1/2^{m+1}, t), \quad \text{we have} \quad \left( \frac{2\pi \log \frac{5}{4}}{4} \right)^{-1} \]

\[ \int_{|z - m| = t} \log \left| \frac{f_n(z, t)}{f_n(x_m, t)} \right| \, dz \]

and hence we get

\[ \varphi_m(t) dt \leq \frac{1}{2\pi \log \frac{5}{4}} \int_{0}^{2v} \left\{ \log \left| \sum_{v=0}^{\infty} x_m^{2v} \right| + \rho \left( \sum_{v=0}^{\infty} x_m^{2v} \right)^2 \right\} \frac{1}{2} \]

\[ Q_m = \left( \frac{2}{\pi} \right)^{1/2} \left\{ \frac{1}{m(E)} \sum_{v=0}^{\infty} x_m^{2v} + \rho \left( \sum_{v=0}^{\infty} x_m^{2v} \right)^2 \right\}^{1/2} \]

We get

\[ \varphi_m(t) dt < \frac{m(E)}{2\pi \log \frac{5}{4}}^{1/2} \left\{ \log V(x_m, \theta) \right\} + \rho \left( \sum_{v=0}^{\infty} x_m^{2v} \right)^2 \]

where

\[ V(x_m, \theta) = \frac{1}{2\pi} \left\{ \left( 1 - \rho \right) \sum_{v=0}^{\infty} x_m^{2v} \right\} + \rho \left( \sum_{v=0}^{\infty} x_m^{2v} \right)^2 \]
Since \(|x_m| < 1\) in \(V(x_m, 0)\), the second term in both the numerator and denominator is constant. Therefore

\[
V(x_m, \theta) < \frac{1}{\rho \left( \sum_{v=0}^{\infty} x_m^v \right)^2} \left[ 1 - (1 - 2^{-m}) ]^p \right] < \frac{1}{F(3)^2} \rho \left( 1 - \left( 1 - \frac{1}{2m} + \frac{5}{2^{m+2}} \right) \right)
\]

Hence we obtain

\[
\int \varphi_{m(t)} dt < C Q m m(E) \log \frac{1}{m(E)}
\]

If \(E\) has no point in common with a set \(G_m\) of measure at most \(Q m / m^2\), taking \(\varepsilon = m^{-2}\)

Consider

\[
I = \int_{E} \frac{1}{M(t) \log M(t)} dt,
\]

where

\[
M(t) \leq \Phi(n) = \frac{\log n \log \log \log n}{\log 2}
\]

Put

\[
E_k = \{ t \in E : M(t) - k \}
\]

Then

\[
E = \bigcup_{k=m_0}^{\Phi(n)} E_k
\]

and

\[
I = \sum_{k=m_0}^{\Phi(n)} \int_{k \log k}^{t} \frac{\varphi_{m(t)}}{k} dt, \quad \sum_1 + \sum_2.
\]

where \(\sum_1\) contains the terms for which \(m(E_k) \leq m(E) / k^2\).

First consider \(\sum_1\). The function \(x \log x^{-1}\) is increasing with \(x\) for \(0 < x < e^{-1}\) and therefore

\[
m(E_k) \log \frac{1}{m(E_k)} \leq 2 m(E) \log \frac{k^2}{k} + \frac{1}{k^2} m(E) \log \frac{1}{m(E)}
\]

If \(\frac{m(E_k)}{k^2} < \frac{1}{e} \) or \(m \varepsilon^2 > e m(E)\). Now
\[
\sum_{m_0 \leq m \leq k} \frac{1}{m^2} \leq \frac{2}{\pi} \left( \frac{\sigma^2}{\sigma_n^2} \right)
\]

If \( E_k \) has no point in common with a set \( H_k \) of measure at most

\[
H_k = \bigcup_{m=m_0}^{k} G_m
\]

and \( P \sum_{m=m_0}^{k} \frac{1}{m^2} \) where \( Pk = \max_{m_0 \leq m \leq k} Q_m \). Now consider \( \sum_{m=m_0}^{k} \), where \( m(E_k) \leq m(E)/k^2 \).

Then

\[
\sum_{m=m_0}^{k} \frac{1}{m^2} \leq \frac{2}{\pi} \left( \frac{\sigma^2}{\sigma_n^2} \right) + (1 - \rho) \sum_{v=0}^{\infty} x_m^{2v} + \rho \left( \sum_{v=0}^{\infty} x_m^{v} \right)^2
\]

Now

\[
Q^2_m = \left( \frac{2}{\pi} \right) \left( \frac{\sigma^2}{\sigma_n^2} \right)
\]

If \( E \) has no point in common with a set \( H \) of measure at most

\[
\left( \frac{1}{m_0} \right) \max_{m_0 \leq m \leq k} Q_m.
\]

Hence

\[
\frac{\sum_{m_0 \leq m \leq k} \left( \frac{P_k m(E_k)}{K^2} + \frac{P_k}{K^2 \log k} m(E) \log \frac{1}{m(E)} \right)}{K^2 \log k}
\]

since the second term in both the numerator and denominator is dominant. Therefore
Therefore we have

\[ P_k < \left( \frac{4}{\pi} \right)^{1/2} \left[ 1 - (1 - 2^{-\Phi(n)})^{n+1} \right]^{-1} \]

\[ \leq \left( \frac{4}{\pi} \right)^{1/2} \left[ 1 - (\log n_0)^{-1} \right]^{-1} \]

\[ < \left( \frac{4}{\pi} \right)^{1/2} e^2 \text{if } n_0 > e^{1/\sqrt{2}} \]

Hence after solving the theorem and lemmas we have conclude that considering a polynomial (1.1) we have estimate bounds of the number of level crossings of the above random algebraic polynomials where under a given condition with mean zero and joint density function

\[ |M|^{1/2} (2\pi)^{-n/2} \exp\left[ (-1/2)\delta'M\delta \right] \].

There exists an integer \( n_0 \) and a set \( E \) of measure at most \( A/\left( \log n_0 - \log \log \log n_0 \right) \) such that, for each \( n > n_0 \) and all not belonging to \( E \), the equations (1.1) satisfying the condition (1.2), have at most \( \alpha (\log \log n)^2 \log n \) roots where \( \alpha \) and \( A \) are constants.

Hence the theorem.

References


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