The Comparison of the convergence rate with different
preconditioners for Linear Systems
Aijuan Li

School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, PR China
juanzi612@163.com

Abstract
In this paper, the preconditioned Gauss-Seidel iterative methods are proposed with different
preconditioners. The comparison theorem is obtained under the different preconditioners when the
coefficient matrix A of linear system is a nonsingular M− matrix. This generalizes the result in [1].
Numerical example are given to illustrate our theoretical result.

Keywords: Gauss-Seidel iterative, spectral radius, M-matrix, preconditioner

1 Introduction
We consider the linear system of n equations

$$Ax = b$$ (1)

Where $$A = (a_{ij}) \in R^{n \times n}$$ and $$b \in R^n$$ are given and $$x \in R^n$$ is unknown.
Assume that

$$A = M - N$$

Where M is nonsingular. Then the basic iterative method for solving (1) can be expressed in the
form

$$x^{(k+1)} = M^{-1}N x^{(k)} + M^{-1}b, k = 0,1,\ldots$$

Where $$x^{(0)}$$ is an initial vector. As it is well known, the above iterative process is convergent to the
unique solution $$x = A^{-1}b$$ for each initial value $$x^{(0)}$$ if and only if the spectral radius of the
iteration matrix $$M^{-1}N$$ satisfies $$\rho(M^{-1}N) < 1$$.

For simplicity, we let $$A = I - L - U$$, where I is the identity matrix, L and U are strictly
lower and strictly upper triangular matrices, respectively. Then the iteration matrix of the
Gauss-Seidel iterative method for solving the linear system (1) is

\[ T = (I - L)^{-1}U \quad (2) \]

In order to accelerate the convergence of iterative method for solving the linear system (1), the original system (1) is transformed into the following preconditioned linear system

\[ PAx = Pb \quad (3) \]

where \( P \in \mathbb{R}^{nxn} \) is nonsingular and called a preconditioner. Then the corresponding basic iterative method is given in general by

\[ x^{(k+1)} = M_p^{-1}N_p x^{(k)} + M_p^{-1}Pb, k = 0, 1, 2 \ldots \]

where \( PA = M_p - N_p \) is a splitting of \( PA \) and \( M_p \) is nonsingular. Similar to the original system (1), we call the basic iterative methods corresponding to the preconditioned system the preconditioned iterative methods, such as the preconditioned Gauss-Seidel method and preconditioned AOR iterative method.

In [1]-[9], some different preconditioners have been proposed by several authors. In [1], the author presented preconditioned Gauss-Seidel method for linear systems and compared the convergence rate by using different preconditioners.

In this paper, we propose the new preconditioned Gauss-Seidel with the preconditoners \( P_1 \) and \( P_2 \), respectively. Furthermore, we compare the convergence rate of preconditioned Gauss-Seidel with the preconditoners \( P_1 \) and \( P_2 \).

The preconditioner \( P_1 \) is of the form \( P_1 = I + R_\alpha + U \), where

\[
R_\alpha = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
-\alpha_1 a_{n1} & -\alpha_2 a_{n2} & \cdots & -\alpha_{n-1} a_{nn-1} & 0
\end{pmatrix}
\]
and \( \alpha_i(i=1,2,\cdots,n-1) \) are real numbers. If \( \alpha_i=1(i=1,2,\cdots,n-1) \), the \( R_{\alpha} \) becomes \( R \) in [1].

The preconditioner \( P_2 \) is of the form \( P_2 = I + R_{\alpha} + S \), where

\[
S = \begin{pmatrix}
0 & -a_{12} & 0 & \cdots & 0 \\
0 & 0 & -a_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -a_{n-1n}
\end{pmatrix}
\]

If \( \alpha_i=1(i=1,2,\cdots,n-1) \), the preconditioners \( P_1 \) and \( P_2 \) become the preconditioners \( P_{RU} \) and \( P_{SR} \), respectively.

For convenience, some notations, definitions, lemmas and the theorems that will be used in the following parts are given below.

II Preliminaries

In this paper, \( \rho(\cdot) \) denotes the spectral radius of a matrix.

**Definition 2.1([10]).** For \( A=(a_{ij}), B= (b_{ij}) \in R^{m \times n} \), we write \( A \geq B \), if \( a_{ij} \geq b_{ij} \) holds for all \( i, j=1,2,\cdots,n \). Calling \( A \) nonnegative matrix if \( A \geq 0 \) and \( a_{ij} \geq 0 \) for all \( i, j=1,2,\cdots,n \).

**Definition 2.2([11]).** A matrix \( A \) is a L-matrix if \( a_{ij} \geq 0, i=1,2,\cdots,n \) and \( a_{ij} \leq 0 \) for all \( i, j=1,2,\cdots,n, \ i \neq j \). A nonsingular L-matrix \( A \) is a nonsingular M-matrix if \( A^{-1} \geq 0 \).

**Lemma 2.1([10]).** Let \( A \) be a nonnegative \( n \times n \) nonzero matrix. Then

(a) \( \rho(A) \), the spectral radius of \( A \), is an eigenvalue;

(b) \( A \) has a nonnegative eigenvector corresponding to \( \rho(A) \);

(c) \( \rho(A) \) is a simple eigenvalue of \( A \);

(d) \( \rho(A) \) increases when any entry of \( A \) increases.

**Definition 2.3([10]).** For \( n \times n \) real matrices \( A, M \) and \( N, A=M-N \) is a regular splitting of the matrix \( A \) if \( M \) is nonsingular with \( M^{-1} \geq 0 \) and \( N \geq 0 \). Similarly, \( A=M-N \) is a weak regular splitting of the matrix \( A \) if \( M \) is nonsingular with \( M^{-1} \geq 0 \).
and \( M^{-1}N \geq 0 \).

**Lemma 2.2([2]).** Let \( A \) be a nonnegative matrix. Then

1. If \( \alpha x \leq Ax \) for some nonnegative vector \( x, x \neq 0 \), then \( \alpha \leq \rho(A) \).

2. If \( Ax \leq \beta x \) for some positive vector \( x \), then \( \rho(A) \leq \beta \). Moreover, if \( A \) is irreducible and if \( 0 \neq \alpha x \leq Ax \leq \beta x, Ax \neq \beta x \) for some nonnegative vector \( x \), then \( \alpha < \rho(A) < \beta \) and \( x \) is a positive vector.

**Lemma 2.3([1]).** Suppose that \( A_1 = M_1 - N_1 \) and \( A_2 = M_2 - N_2 \) are weak regular splitting of the monotone matrices \( A_1 \) and \( A_2 \), respectively, such that \( M_2^{-1} \geq M_1^{-1} \). If there exists a positive vector \( x \) such that \( 0 \leq A_1x \leq A_2x \). Then, for the monotonic norm associated with \( x \),

\[
\|M_2^{-1}N_2\|_1 \leq \|M_1^{-1}N_1\|_1.
\]

In particular, if \( M_1^{-1}N_1 \) has a positive Perron vector, then

\[
\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1).
\]

**III Preconditioned Gauss-Seidel iterative method and comparison theorem**

For the linear system (1), we consider its preconditioned from

\[
A_1x = P_1Ax = P_1b \quad (4)
\]

where \( P_1 = I + R_\alpha + U \).

Now, we express the coefficient matrix of (4) as

\[
A_1 = P_1A = (I + R_\alpha + U)(I - L - U)
= I - L - U + R_\alpha - R_\alpha L - R_\alpha U + U - UL - U^2
= I - D_0 - D_1 - (L - R_\alpha + R_\alpha L + E_1 + E_0) - (F_0 + U^2)
= M_{UR_\alpha} - N_{UR_\alpha}
\]

where \( UL = D_0 + E_0 + F_0 \) and \( R_\alpha U = D_1 + E_1 \). \( D_0, E_0, \) and \( F_0 \) are diagonal, strictly lower and upper triangular parts of \( UL \), respectively. \( D_1 \) and \( E_1 \) are diagonal and strictly lower triangular parts of \( R_\alpha U \).
Suppose that \( M_{UR_a} = I - D_0 - D_1 - (L - R_a + R_aL + E_1 + E_0) \) \hspace{1cm} \begin{align*} N_{UR_a} &= F_0 + U^2 \end{align*} \hspace{1cm} \begin{align*} (5) \end{align*} \hspace{1cm} \begin{align*} (6) \end{align*} \hspace{1cm} \begin{align*} \end{align*}

Then the perconditioned Gauss-Seidel iteration matrix with the preconditioner \( P \) is known as:

\[ T_{UR_a} = M_{UR_a}^{-1} N_{UR_a} = [(I - D_0 - D_1) - (L - R_a + R_aL + E_1 + E_0)]^{-1}(F_0 + U^2) \hspace{1cm} \begin{align*} (7) \end{align*} \hspace{1cm} \begin{align*} \end{align*}

Similarly, we consider its preconditioned form

\[ A_2x = P_2Ax = P_2b \hspace{1cm} \begin{align*} (8) \end{align*} \hspace{1cm} \begin{align*} \end{align*}

where \( P_2 = I + R_a + S \).

We express the coefficient matrix of (8) as

\[ A_2 = P_2A = (I + R_a + S)(I - L - U) \]
\[ = I - L - U + R_a - R_aL - R_aU + S - SL - SU \]
\[ = I - D_1 - D_2 - (L - R_a + R_aL + E_1 + E_2) - (U - S + SU) \]
\[ = M_{SR_a} - N_{SR_a} \]

Where \( SL = D_2 + E_2 \), \( D_2 \) and \( E_2 \) are diagonal and strictly lower triangular parts of \( SL \), respectively.

Suppose that

\[ M_{SR_a} = I - D_1 - D_2 - (L - R_a + R_aL + E_1 + E_2) \hspace{1cm} \begin{align*} (9) \end{align*} \hspace{1cm} \begin{align*} \end{align*}

\[ N_{SR_a} = U - S - SU \hspace{1cm} \begin{align*} (10) \end{align*} \hspace{1cm} \begin{align*} \end{align*}

Then the preconditioned Gauss-Seidel iteration matrix with the preconditioner \( P_2 \) is known as:

\[ T_{SR_a} = M_{SR_a}^{-1} N_{SR_a} = [(I - D_1 - D_2) - (L - R_a + R_aL + E_1 + E_2)]^{-1}(U - S + SU) \hspace{1cm} \begin{align*} (11) \end{align*} \hspace{1cm} \begin{align*} \end{align*}

**Theorem 3.1** Let \( A_1 \) and \( A_2 \) be the coefficient matrices of linear system (4) and (8), respectively. \( M_{UR_a}, N_{UR_a}, M_{SR_a} \) and \( N_{SR_a} \) are defined by (5),(6),(9) and (10), respectively. Let \( A \) be a nonsingular \( M \)-matrix. Suppose that \[ 0 \leq \sum_{j=k+1}^{n} a_{ij} a_{jk} < 1 \], \[ 0 \leq \sum_{i=1}^{n-1} a_{ii} a_{jm} a_{im} < 1 \] and \[ 0 \leq \alpha_i \leq 1 \] for \( i = 1, 2, \cdots, n - 1 \). Then \( A_1 = M_{UR_a} - N_{UR_a} \) and \( A_2 = M_{SR_a} - N_{SR_a} \) are weak regular splitting of \( A_1 \) and \( A_2 \), respectively.
Proof. First, we prove that $A_i = M_{\text{UR}_{\alpha}} - N_{\text{UR}_{\alpha}}$ is weak regular splitting of $A_i$. Since $A$ is nonsingular $M$-matrix, $0 \leq \sum_{j=k+1}^{n} a_{ij} a_{jk} < 1$, $0 \leq \sum_{i=1}^{n} \alpha_i a_{ni} a_{in} < 1$ and $0 \leq \alpha_i \leq 1$,

$$M_{\text{UR}_{\alpha}}^{-1} = [(I - D_0 - D_1) - (L - R_{\alpha} + R_{\alpha} L + E_1 + E_0)]^{-1} = [(I - (I - D_0 - D_1))^{-1}(L - R_{\alpha} + R_{\alpha} L + E_1 + E_0)]^{-1}(I - D_0 - D_1)^{-1} = [(I + (I - D_0 - D_1))^{-1}(L - R_{\alpha} + R_{\alpha} L + E_1 + E_0) + [(I - D_0 - D_1)^{-1}(L - R_{\alpha} + R_{\alpha} L + E_1 + E_0)]^2 + \cdots] (I - D_0 - D_1)^{-1} \geq 0$$

We know that $N_{\text{UR}_{\alpha}} = F_0 + U^2 \geq 0$. Therefore, $M_{\text{UR}_{\alpha}}^{-1} N_{\text{UR}_{\alpha}} \geq 0$. By Definition 2.3, we obtain that $A_i = M_{\text{UR}_{\alpha}} - N_{\text{UR}_{\alpha}}$ is weak regular splitting of $A_i$.

Now, we will prove that $A_j = M_{\text{SR}_{\alpha}} - N_{\text{SR}_{\alpha}}$ is weak regular splitting of $A_j$.

Since $A$ is a nonsingular $M$-matrix, we have $0 \leq a_{ij} a_{ji} < 1$ for $i, j = 1, 2, \ldots, n - 1$. According to $0 \leq \sum_{i=1}^{n} \alpha_i a_{ni} a_{in} < 1$ and $0 \leq \alpha_i \leq 1$ ($i = 1, 2, \ldots, n - 1$), we obtain that

$$M_{\text{SR}_{\alpha}}^{-1} = [(I - D_0 - D_2) - (L - R_{\alpha} + R_{\alpha} L + E_1 + E_2)]^{-1} = [(I - (I - D_0 - D_2))^{-1}(L - R_{\alpha} + R_{\alpha} L + E_1 + E_2)]^{-1}(I - D_0 - D_2)^{-1} = [(I + (I - D_0 - D_2))^{-1}(L - R_{\alpha} + R_{\alpha} L + E_1 + E_2) + [(I - D_0 - D_2)^{-1}(L - R_{\alpha} + R_{\alpha} L + E_1 + E_2)]^2 + \cdots] (I - D_0 - D_2)^{-1} \geq 0$$

We know that $N_{\text{SR}_{\alpha}} = U - S + SU \geq 0$. By Definition 2.3, we obtain that $A_j = M_{\text{SR}_{\alpha}} - N_{\text{SR}_{\alpha}}$ is weak regular splitting of $A_j$. This completes the proof.

Theorem 3.2 Let $A_1$ and $A_2$ be the coefficient matrices of linear system (4) and (8), respectively. $M_{\text{UR}_{\alpha}}$, $N_{\text{UR}_{\alpha}}$, $M_{\text{SR}_{\alpha}}$ and $N_{\text{SR}_{\alpha}}$ are defined by (5),(6),(9) and (10), respectively. Let $A$ be a nonsingular $M$-matrix. Suppose that $0 \leq \sum_{j=k+1}^{n} a_{ij} a_{jk} < 1$, $0 \leq \sum_{i=1}^{n} \alpha_i a_{ni} a_{in} < 1$ and $0 \leq \alpha_i \leq 1$ for $i, j = 1, 2, \ldots, n - 1$. Then $\rho (M_{\text{UR}_{\alpha}}^{-1} N_{\text{UR}_{\alpha}}) \leq \rho (M_{\text{SR}_{\alpha}}^{-1} N_{\text{SR}_{\alpha}})$.

Proof. For a positive vector $x$ and $A$ is a nonsingular $M$-matrix,
\[ A_i x = (I + R_{a_i} + U) A x \geq (I + R_{a_i} + S) A x \geq 0. \] We have

\[
\begin{aligned}
M_{SR_{a_i}}^{-1} - M_{UR_{a_i}}^{-1} &= (I - D_1 - D_2) - (L - R_{a_i} L + E_1 + E_2) \\
- [(I - D_0 - D_2) - (L - R_{a_i} L + E_1 + E_0)] \\
&= (D_0 + E_0) - (D_2 + E_2) \\
&= (D_0 + E_0) - SL \geq 0
\end{aligned}
\]  

(12) By Theorem 3.1, we know that \( M_{UR_{a_i}}^{-1} \geq 0 \) and \( M_{SR_{a_i}}^{-1} \geq 0 \). Pre-multiplying and post-multiplying (12) by \( M_{UR_{a_i}}^{-1} \) and \( M_{SR_{a_i}}^{-1} \), respectively, we have

\[
M_{UR_{a_i}}^{-1} - M_{SR_{a_i}}^{-1} \geq 0
\]

Thus, \( M_{UR_{a_i}}^{-1} \geq M_{SR_{a_i}}^{-1} \). By Lemma 2.3 and Theorem 3.1, we obtain that

\[
\rho \left( M_{UR_{a_i}}^{-1} N_{UR_{a_i}} \right) \leq \rho \left( M_{SR_{a_i}}^{-1} N_{SR_{a_i}} \right)
\]

This completes the proof.

**Remark** If \( \alpha_i = 1 \) for \( i = 1, 2, \ldots, n - 1 \), Theorem 3.2 becomes the result of Theorem 4.3 in [1].

**IV Numerical example**

In this section, we give the following example to illustrate the results obtained in section 3.

**Example** The coefficient matrix \( A \) of (1) is given by

\[
A = \begin{pmatrix}
1 & -0.2 & -0.3 & -0.1 & -0.2 \\
-0.1 & 1 & -0.1 & -0.3 & -0.1 \\
-0.2 & -0.1 & 1 & -0.1 & -0.2 \\
-0.2 & -0.1 & -0.1 & 1 & -0.3 \\
-0.1 & -0.2 & -0.2 & -0.1 & 1
\end{pmatrix}
\]

We see that \( A \) satisfies the condition of Theorem 3.1 and Theorem 3.2.

If \( \alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.2, \alpha_4 = 1 \), we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners \( P_1 \) and \( P_2 \) by \( \rho \left( M_{UR_{a_i}}^{-1} N_{UR_{a_i}} \right) \) and \( \rho \left( M_{SR_{a_i}}^{-1} N_{SR_{a_i}} \right) \), respectively.

If \( \alpha_1 = 0.2, \alpha_2 = 0.5, \alpha_3 = 1, \alpha_4 = 0.2 \), we denote the spectral radius of the preconditioned
Gauss-Seidel iterative matrix with the preconditioners \( P_1 \) and \( P_2 \) by \( \rho \left( M_{U,R_0}^{-1} N_{U,R_0} \right) \) and 
\[ \rho \left( M_{S_{i,R_0}}^{-1} N_{S_{i,R_0}} \right), \]
respectively.

If \( \alpha_1 = 0.8 \), \( \alpha_2 = 0.2 \), \( \alpha_3 = 0.3 \), \( \alpha_4 = 0.5 \), we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners \( P_1 \) and \( P_2 \) by \( \rho \left( M_{U,R_0}^{-1} N_{U,R_0} \right) \) and 
\[ \rho \left( M_{S_{i,R_0}}^{-1} N_{S_{i,R_0}} \right), \]
respectively.

If \( \alpha_1 = 0.1 \), \( \alpha_2 = 1 \), \( \alpha_3 = 1 \), \( \alpha_4 = 1 \), we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners \( P_1 \) and \( P_2 \) by \( \rho \left( M_{U,R_0}^{-1} N_{U,R_0} \right) \) and 
\[ \rho \left( M_{S_{i,R_0}}^{-1} N_{S_{i,R_0}} \right), \]
respectively.

If \( \alpha_1 = 0.9 \), \( \alpha_2 = 0.4 \), \( \alpha_3 = 0.8 \), \( \alpha_4 = 0.5 \), we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners \( P_1 \) and \( P_2 \) by \( \rho \left( M_{U,R_0}^{-1} N_{U,R_0} \right) \) and 
\[ \rho \left( M_{S_{i,R_0}}^{-1} N_{S_{i,R_0}} \right), \]
respectively. Then we obtain the Table 1.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \rho \left( M_{U,R_0}^{-1} N_{U,R_0} \right) )</th>
<th>( \rho \left( M_{S_{i,R_0}}^{-1} N_{S_{i,R_0}} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1818</td>
<td>0.3313</td>
</tr>
<tr>
<td>2</td>
<td>0.1745</td>
<td>0.3110</td>
</tr>
<tr>
<td>3</td>
<td>0.1732</td>
<td>0.3137</td>
</tr>
<tr>
<td>4</td>
<td>0.1570</td>
<td>0.2724</td>
</tr>
<tr>
<td>5</td>
<td>0.1670</td>
<td>0.3002</td>
</tr>
</tbody>
</table>

From Table 1, we can see that 
\[ \rho \left( M_{U,R_0}^{-1} N_{U,R_0} \right) \leq \rho \left( M_{S_{i,R_0}}^{-1} N_{S_{i,R_0}} \right). \]

**Conjectures** In this paper, the preconditioners \( P_1 \) and \( P_2 \) are generalized to the preconditioners with multi-parameters, the result may be correct.
References


